

# LATTICE EQUATIONS ARISING FROM DISCRETE PAINLEVÉ SYSTEMS

## (I): $(A_2 + A_1)^{(1)}$ AND $(A_1 + A'_1)^{(1)}$ CASES

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**ABSTRACT.** We introduce the concept of  $\omega$ -lattice, constructed from  $\tau$  functions of Painlevé systems, on which quad-equations of ABS type appear. In particular, we consider the  $A_5^{(1)}$ - and  $A_6^{(1)}$ -surface  $q$ -Painlevé systems corresponding affine Weyl group symmetries are of  $(A_2 + A_1)^{(1)}$ - and  $(A_1 + A_1)^{(1)}$ -types, respectively.

### 1. INTRODUCTION

**1.1. Main result.** Although two important and widely used classifications of integrable discrete systems have been known for more than a decade, no satisfactory relation between the two is yet understood. The first is the ABS classification of integrable partial difference equations [1–5] while the second is Sakai’s classification of integrable nonlinear ordinary difference equations [30]. Using geometry and symmetry groups of the equations as our main tool, we present a new, general approach to connect the two classifications.

The framework we describe is based on the  $\omega$ -lattice, which is related to  $\tau$ -function theory. While this framework is general, we explain its construction for the  $A_5^{(1)}$ - and  $A_6^{(1)}$ -surface  $q$ -Painlevé systems and show how the ABS quad-equations appear.

The construction of the  $\omega$ -lattice is essential for knowledge about how ABS-type equations can be reduced to a discrete Painlevé equation. It provides not only the type of equation, but also the combinatorial structure of the lattice before reduction. In [14], we showed how to use this information to find a reduction of equations on a 4-dimensional hypercube (4D cube) but we did not provide details of the  $\omega$ -lattice construction. Subsequently, in [13] we provided a comprehensive method for constructing Lax pairs of the  $A_5^{(1)}$ -surface  $q$ -Painlevé equations. The construction of the  $\omega$ -lattices for  $A_5^{(1)}$ - and  $A_6^{(1)}$ -surface  $q$ -Painlevé systems provided in the present paper leads to the following main result.

**Theorem 1.1.** *All quad-equations appearing on the  $\omega$ -lattice for  $A_5^{(1)}$ -surface  $q$ -Painlevé system, defined by Equation (3.6), and those for  $A_6^{(1)}$ -surface  $q$ -Painlevé system, defined by Equation (4.18), are of ABS type.*

**1.2. Background.** Discrete Painlevé equations and ABS equations have been studied from various viewpoints. In particular, Sakai [30] gave a classification of discrete Painlevé equations based on the geometric structure of rational surfaces, and their corresponding affine Weyl symmetry group. On the other hand, Adler, Bobenko and Suris [1, 2], and later Boll [3–5], showed how to classify quad-equations (partial difference equations on quadrilateral lattices) based on consistency of the equations on 3-dimensional cubes (see Section 1.3). The resulting equations are called ABS equations.

Many types of periodic reductions from ABS equations to discrete Painlevé equations have been investigated [7–10, 14, 24, 26, 27]. It is well known that some discrete Painlevé equations can be derived from ABS equations by periodic reductions with suitable choice

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of dependent variables. However, (i) after applying a periodic reduction to an ABS equation, we do not know which of discrete Painlevé equations appear; (ii) discrete Painlevé equations obtained by periodic reductions often have insufficient number of parameters (such an example appeared in [8], where the reduction which is given by Equations (1.9)–(1.15) for the special value  $\lambda = 1$  is discussed). To solve the above-mentioned problems systematically, it is necessary to study the periodic reductions not only from the viewpoint of ABS equations but also from that of Painlevé systems. Unlike other investigations, which start with a quad-equation and obtain a discrete Painlevé equation, we show how to obtain the reverse, by investigating underlying bilinear structure of  $\tau$  functions for the discrete Painlevé equation.

**1.3. ABS equation.** In [1–5], Adler *et al.* classified polynomials in four variables into eleven types:  $Q4$ ,  $Q3$ ,  $Q2$ ,  $Q1$ ,  $H3$ ,  $H2$ ,  $H1$ ,  $D4$ ,  $D3$ ,  $D2$ ,  $D1$ . The first four types, the next three types and the last four types are collectively called  $Q$ -,  $H^4$ - and  $H^6$ -types, respectively. The resulting polynomial  $P$  satisfies the following properties.

**(1) Linearity:** Polynomial  $P$  is linear in each argument, i.e., it has the following form:

$$P(x_1, x_2, x_3, x_4) = A_1 x_1 x_2 x_3 x_4 + \cdots + A_{16}, \quad (1.1)$$

where coefficients  $A_i$  are complex parameters.

**(2) 3D consistency and tetrahedron property:** There exist seven polynomials in four variables:  $P^{(i)}$ ,  $i = 1, \dots, 7$ , which satisfy the property **(1)** and a cube  $C$  on whose six faces the following quad-equations are assigned

$$P(x_0, x_1, x_2, x_{12}) = 0, \quad P^{(1)}(x_0, x_2, x_3, x_{23}) = 0, \quad (1.2a)$$

$$P^{(2)}(x_0, x_3, x_1, x_{31}) = 0, \quad P^{(3)}(x_3, x_{31}, x_{23}, x_{123}) = 0, \quad (1.2b)$$

$$P^{(4)}(x_1, x_{12}, x_{31}, x_{123}) = 0, \quad P^{(5)}(x_2, x_{23}, x_{12}, x_{123}) = 0, \quad (1.2c)$$

where eight variables  $x_i$  are on the vertices of the cube, such that  $x_{123}$  can be uniquely expressed by the four variables  $x_i$ ,  $i = 0, 1, 2, 3$ , (*3D consistency*) and the following relations hold (*tetrahedron property*):

$$P^{(6)}(x_0, x_{12}, x_{23}, x_{31}) = 0, \quad P^{(7)}(x_1, x_2, x_3, x_{123}) = 0. \quad (1.3)$$

We here list some polynomials of ABS type as follows:

$$Q1 : Q1(x_1, x_2, x_3, x_4; \alpha_1, \alpha_2; \epsilon)$$

$$= \alpha_1(x_1 x_2 + x_3 x_4) - \alpha_2(x_1 x_4 + x_2 x_3) - (\alpha_1 - \alpha_2)(x_1 x_3 + x_2 x_4) + \epsilon \alpha_1 \alpha_2 (\alpha_1 - \alpha_2),$$

$$H3 : H3(x_1, x_2, x_3, x_4; \alpha_1, \alpha_2; \delta; \epsilon)$$

$$= \alpha_1(x_1 x_2 + x_3 x_4) - \alpha_2(x_1 x_4 + x_2 x_3) + (\alpha_1^2 - \alpha_2^2) \left( \delta + \frac{\epsilon}{\alpha_1 \alpha_2} x_2 x_4 \right),$$

$$H1 : H1(x_1, x_2, x_3, x_4; \alpha_1, \alpha_2; \epsilon) = (x_1 - x_3)(x_2 - x_4) + (\alpha_2 - \alpha_1)(1 - \epsilon x_2 x_4),$$

$$D4 : D4(x_1, x_2, x_3, x_4; \delta_1, \delta_2, \delta_3) = x_1 x_3 + x_2 x_4 + \delta_1 x_1 x_4 + \delta_2 x_3 x_4 + \delta_3,$$

where  $\alpha_1, \alpha_2 \in \mathbb{C}^*$  and  $\epsilon, \delta, \delta_1, \delta_2, \delta_3 \in \{0, 1\}$ . It is well known that assigning a polynomial of ABS type to all faces of the integer lattice  $\mathbb{Z}^2$ , we can obtain an integrable partial difference equation, e.g.

**discrete Schwarzian KdV equation [22, 23]:**

$$Q1(U, \overline{U}, \widehat{U}, \widehat{\overline{U}}; \alpha, \beta; 0) = 0 \Leftrightarrow \frac{(U - \overline{U})(\widehat{U} - \widehat{\overline{U}})}{(U - \widehat{U})(\overline{U} - \widehat{\overline{U}})} = \frac{\alpha}{\beta}; \quad (1.4)$$

**lattice modified KdV equation [1, 22, 25]:**

$$H3(U, \bar{U}, -\widehat{\bar{U}}, \widehat{U}; \alpha, \beta; 0; 0) = 0 \Leftrightarrow \frac{\widehat{\bar{U}}}{U} = \frac{\alpha \bar{U} - \beta \widehat{U}}{\alpha \widehat{\bar{U}} - \beta U}; \quad (1.5)$$

**lattice potential KdV equation [12, 22]:**

$$H1(U, \bar{U}, \widehat{\bar{U}}, \widehat{U}; \alpha, \beta; 0) = 0 \Leftrightarrow (U - \widehat{\bar{U}})(\bar{U} - \widehat{U}) = \alpha - \beta; \quad (1.6)$$

**discrete version of Volterra-Kac-van Moerbeke equation [22]:**

$$\begin{aligned} D4(1 - (\alpha^{-1}\beta - 1)U, \widehat{U}, \bar{U}, -1 + (\alpha^{-1}\beta - 1)\widehat{\bar{U}}; 0, 0, 0) &= 0 \\ \Leftrightarrow \frac{\widehat{U}}{\bar{U}} &= \frac{(\beta - \alpha)U - \alpha}{(\beta - \alpha)\widehat{\bar{U}} - \alpha}, \end{aligned} \quad (1.7)$$

where

$$U = U_{l,m}, \quad \alpha = \alpha_l, \quad \beta = \beta_m, \quad \bar{\cdot} : l \rightarrow l+1, \quad \widehat{\cdot} : m \rightarrow m+1, \quad l, m \in \mathbb{Z}. \quad (1.8)$$

Throughout this paper, we often call a partial difference equation by the type of corresponding ABS polynomial.

The relations between ABS equations and discrete Painlevé equations on the level of equations have been intensively investigated, but those on the level of underlying structure have not been clarified. Here, we show an example of such a relation (the special case  $\lambda = 1$  is first obtained in [8]). By letting

$$U_{l,m} = \lambda^l \Omega_{l,m}, \quad (1.9)$$

and applying the  $(1, -2)$ -periodic condition

$$\Omega_{l+1,m-2} = \Omega_{l,m}, \quad (1.10)$$

which implies the condition on the parameters

$$\frac{\bar{\alpha}}{\alpha} = \frac{\beta}{\widehat{\widehat{\beta}}}, \quad (1.11)$$

Equation (1.5) can be reduced to

$$\frac{\widehat{\widehat{\bar{\Omega}}}}{\bar{\Omega}} = \frac{\widehat{\bar{\Omega}} - \lambda \frac{\alpha}{\beta} \widehat{\bar{\Omega}}}{\lambda \left( \lambda \widehat{\bar{\Omega}} - \frac{\alpha}{\beta} \widehat{\bar{\Omega}} \right)}, \quad (1.12)$$

where

$$\Omega = \Omega_{l,m}. \quad (1.13)$$

Substituting

$$f = \lambda \frac{\widehat{\bar{\Omega}}}{\bar{\Omega}}, \quad g = \lambda \frac{\widehat{\bar{\Omega}}}{\bar{\Omega}}, \quad t = -\frac{\alpha}{\beta}, \quad a = \frac{\beta}{\bar{\beta}}, \quad q = \frac{\bar{\alpha}}{\alpha} = \frac{\beta}{\widehat{\widehat{\beta}}}, \quad (1.14)$$

in Equation (1.12), we obtain the  $A_5^{(1)}$ -surface  $q$ -Painlevé equation known as a  $q$ -discrete analogue of Painlevé III equation (denoted by  $q$ -P<sub>III</sub>) [20, 30]:

$$\bar{g} = \frac{\lambda^2}{gf} \frac{1+tf}{t+f}, \quad \bar{f} = \frac{\lambda^2}{f\bar{g}} \frac{1+at\bar{g}}{at+\bar{g}}. \quad (1.15)$$

**1.4. Plan of the paper.** This paper is organized as follows: in Section 2, we introduce the  $\tau$  functions of  $A_5^{(1)}$ -surface  $q$ -Painlevé systems, which have the extended affine Weyl group symmetry of type  $(A_2 + A_1)^{(1)}$ . Moreover, we show that  $q$ -Painlevé equations can be derived from a birational representation of the extended affine Weyl group of type  $(A_2 + A_1)^{(1)}$ . In Section 3, we construct a lattice where quad-equations appear, and then derive various quad-equations of ABS type, as relations on the lattice. In Sections 4, we summarize the result for the case of  $A_6^{(1)}$ -surface  $q$ -Painlevé systems. Some concluding remarks are given in Section 5.

## 2. CONSTRUCTION OF LATTICES FROM AFFINE WEYL GROUP $\widetilde{W}((A_2 + A_1)^{(1)})$

In this section, we describe 3-dimensional structures constructed by using the symmetry groups of discrete Painlevé equations. While the groups themselves are well known, the novel perspective we focus on is the construction of 3-dimensional lattices based on  $\tau$  functions and  $q$ -Painlevé systems.

**2.1. The  $\tau$ -lattice.** We describe the action of the family of Bäcklund transformations of  $q$ -P<sub>III</sub> (1.15) on six particular variables associated with this system [31]. Iterating these variables under the affine Weyl group actions, we obtain a system of  $\tau$  functions, which form a  $\tau$ -lattice.

The transformation group  $\widetilde{W}((A_2 + A_1)^{(1)})$  has 7 generators  $s_0, s_1, s_2, \pi, w_0, w_1, r$ . Below, we describe their actions on parameters:  $a_0, a_1, a_2, c$ , and on variables:  $\tau_i, \bar{\tau}_i, i = 0, 1, 2$ . Actions on parameters are given by

$$\begin{aligned} s_i &: (a_i, a_{i+1}, a_{i+2}, c) \rightarrow (a_i^{-1}, a_i a_{i+1}, a_i a_{i+2}, c), & \pi &: (a_0, a_1, a_2, c) \rightarrow (a_1, a_2, a_0, c), \\ w_0 &: (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, c^{-1}), & w_1 &: (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, q^{-2}c^{-1}), \\ r &: (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, q^{-1}c^{-1}), \end{aligned}$$

while its actions on variables are given by

$$\begin{aligned} s_i(\tau_i) &= \frac{u_i \tau_{i+1} \bar{\tau}_{i-1} + \bar{\tau}_{i+1} \tau_{i-1}}{u_i^{1/2} \bar{\tau}_i}, & s_i(\tau_j) &= \tau_j \quad (i \neq j), \\ s_i(\bar{\tau}_i) &= \frac{v_i \bar{\tau}_{i+1} \tau_{i-1} + \tau_{i+1} \bar{\tau}_{i-1}}{v_i^{1/2} \tau_i}, & s_i(\bar{\tau}_j) &= \bar{\tau}_j \quad (i \neq j), \\ \pi(\tau_i) &= \tau_{i+1}, & \pi(\bar{\tau}_i) &= \bar{\tau}_{i+1}, \\ w_0(\bar{\tau}_i) &= \frac{a_{i+1}^{1/3} (\bar{\tau}_i \tau_{i+1} \tau_{i+2} + u_{i-1} \tau_i \bar{\tau}_{i+1} \tau_{i+2} + u_{i+1}^{-1} \tau_i \tau_{i+1} \bar{\tau}_{i+2})}{a_{i+2}^{1/3} \bar{\tau}_{i+1} \bar{\tau}_{i+2}}, & w_0(\tau_i) &= \tau_i, \\ w_1(\tau_i) &= \frac{a_{i+1}^{1/3} (\tau_i \bar{\tau}_{i+1} \bar{\tau}_{i+2} + v_{i-1} \bar{\tau}_i \tau_{i+1} \bar{\tau}_{i+2} + v_{i+1}^{-1} \bar{\tau}_i \bar{\tau}_{i+1} \tau_{i+2})}{a_{i+2}^{1/3} \tau_{i+1} \tau_{i+2}}, & w_1(\bar{\tau}_i) &= \bar{\tau}_i, \\ r(\tau_i) &= \bar{\tau}_i, & r(\bar{\tau}_i) &= \tau_i, \end{aligned}$$

where

$$u_i = q^{-1/3} c^{-2/3} a_i, \quad v_i = q^{1/3} c^{2/3} a_i, \quad q = a_0 a_1 a_2, \quad (2.1)$$

and  $i, j \in \mathbb{Z}/3\mathbb{Z}$ . For each element  $w \in \widetilde{W}((A_2 + A_1)^{(1)})$  and function  $F = F(a_i, c, \tau_j, \bar{\tau}_k)$ , we use the notation  $w.F$  to mean  $w.F = F(w.a_i, w.c, w.\tau_j, w.\bar{\tau}_k)$ , that is,  $w$  acts on the arguments from the left.

**Remark 2.1.** Notations in this paper are related to those in [31] by the following correspondence:

$$\begin{aligned} (s_0, s_1, s_2, \pi, w_0, w_1, r) &\rightarrow (s_0, s_1, s_2, \pi^2, r_1, r_0, \pi^3), \\ (a_0, a_1, a_2, c) &\rightarrow (a_0, a_1, a_2, q^{-1}b_0), \\ (\tau_0, \tau_1, \tau_2, \bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2) &\rightarrow (\tau_3, \tau_1, \tau_5, \tau_6, \tau_4, \tau_2). \end{aligned}$$

We also note that in [31] each element  $w \in \widetilde{W}((A_2 + A_1)^{(1)})$  acts on the arguments from the right, whereas in the present paper it acts from the left.

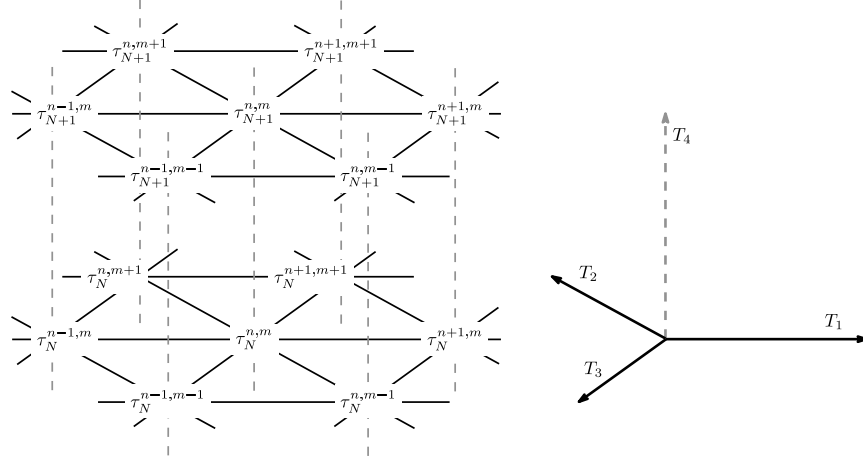


Figure 1. Configuration of  $\tau$  functions on the  $\tau$ -lattice.  $\tau$  functions are defined on the intersections of four lines.

The following proposition shows that  $\widetilde{W}((A_2 + A_1)^{(1)})$  gives a representation of an extended affine Weyl group of type  $(A_2 + A_1)^{(1)}$ .

**Proposition 2.2** ([31]). *The group of transformations  $\widetilde{W}((A_2 + A_1)^{(1)}) = \langle s_0, s_1, s_2, \pi, w_0, w_1, r \rangle$  forms the extended affine Weyl group of type  $(A_2 + A_1)^{(1)}$ . Namely, the transformations satisfy the fundamental relations*

$$s_i^2 = (s_i s_{i+1})^3 = \pi^3 = 1, \quad \pi s_i = s_{i+1} \pi, \quad (i \in \mathbb{Z}/3\mathbb{Z}), \quad (2.2a)$$

$$w_0^2 = w_1^2 = r^2 = 1, \quad r w_0 = w_1 r, \quad (2.2b)$$

and the action of  $\widetilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$  and that of  $\widetilde{W}(A_1^{(1)}) = \langle w_0, w_1, r \rangle$  commute. Note that the parameters  $q$  and  $c$  are invariant under the action of  $\widetilde{W}((A_2 + A_1)^{(1)})$  and  $\widetilde{W}(A_2^{(1)})$ , respectively.

To iterate each variable  $\tau_i$ ,  $\bar{\tau}_i$ , we need the following translations  $T_i$ ,  $i = 1, 2, 3, 4$ , defined by

$$T_1 = \pi s_2 s_1, \quad T_2 = \pi s_0 s_2, \quad T_3 = \pi s_1 s_0, \quad T_4 = r w_0. \quad (2.3)$$

The actions of these on the parameters are given by

$$T_1 : (a_0, a_1, a_2, c) \rightarrow (q a_0, q^{-1} a_1, a_2, c), \quad (2.4a)$$

$$T_2 : (a_0, a_1, a_2, c) \rightarrow (a_0, q a_1, q^{-1} a_2, c), \quad (2.4b)$$

$$T_3 : (a_0, a_1, a_2, c) \rightarrow (q^{-1} a_0, a_1, q a_2, c), \quad (2.4c)$$

$$T_4 : (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, q c). \quad (2.4d)$$

Note that  $T_i$ ,  $i = 1, 2, 3, 4$ , commute with each other and  $T_1 T_2 T_3 = 1$ . We define  $\tau$  functions by

$$\tau_N^{n,m} = T_1^n T_2^m T_4^N(\tau_1), \quad (2.5)$$

where  $n, m, N \in \mathbb{Z}$  and the  $\tau$ -lattice is as shown in Figure 1. We note that

$$\tau_0 = \tau_0^{-1,0}, \quad \tau_1 = \tau_0^{0,0}, \quad \tau_2 = \tau_0^{0,1}, \quad \bar{\tau}_0 = \tau_1^{-1,0}, \quad \bar{\tau}_1 = \tau_1^{0,0}, \quad \bar{\tau}_2 = \tau_1^{0,1}. \quad (2.6)$$

**Remark 2.3.** By definition, action of  $\widetilde{W}((A_2 + A_1)^{(1)})$  gives the relations of points on  $\tau$ -lattice (bilinear equations) and any point of  $\tau$ -lattice (or  $\tau$  function) is determined by six initial points:  $\tau_i, \bar{\tau}_i$ ,  $i = 0, 1, 2$ .

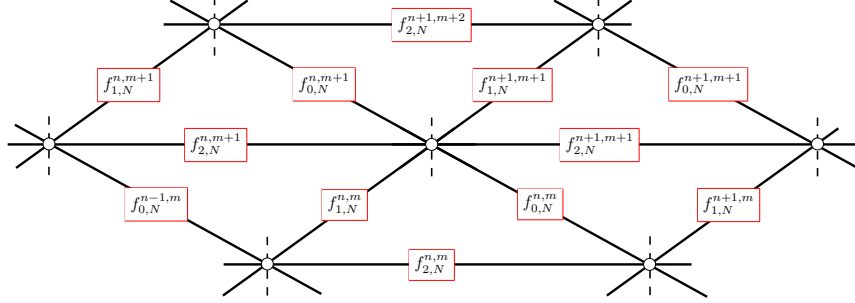


Figure 2. Configuration of  $f$ -functions on the  $f$ -lattice.  $f$ -functions are defined on the edges of the triangle lattices.

**2.2. The discrete Painlevé lattice.** In this section, we construct a 3-dimensional lattice that relates ratios of  $\tau$  functions. The ratios, defined in (2.7), turn out to satisfy a rich set of relations, which give rise not only to  $q$ -P<sub>III</sub> (1.15), but also to  $q$ -P<sub>IV</sub> (2.13) and  $q$ -P<sub>II</sub> (2.21) [18, 19]. We derive these  $q$ -Painlevé equations as relations on the 3-dimensional lattice.

The key starting point is the definition of the following ratios

$$f_0 = q^{1/3} c^{2/3} \frac{\bar{\tau}_1 \tau_2}{\tau_1 \bar{\tau}_2}, \quad f_1 = q^{1/3} c^{2/3} \frac{\bar{\tau}_2 \tau_0}{\tau_2 \bar{\tau}_0}, \quad f_2 = q^{1/3} c^{2/3} \frac{\bar{\tau}_0 \tau_1}{\tau_0 \bar{\tau}_1}, \quad (2.7)$$

where

$$f_0 f_1 f_2 = q c^2. \quad (2.8)$$

The action of  $\tilde{W}((A_2 + A_1)^{(1)})$  on the variables  $f_i$  is given by

$$\begin{aligned} s_i(f_{i-1}) &= f_{i-1} \frac{1 + a_i f_i}{a_i + f_i}, \quad s_i(f_i) = f_i, \quad s_i(f_{i+1}) = f_{i+1} \frac{a_i + f_i}{1 + a_i f_i}, \quad \pi(f_i) = f_{i+1}, \\ w_0(f_i) &= \frac{a_i a_{i+1} (a_{i-1} a_i + a_{i-1} f_i + f_{i-1} f_i)}{f_{i-1} (a_i a_{i+1} + a_i f_{i+1} + f_i f_{i+1})}, \\ w_1(f_i) &= \frac{1 + a_i f_i + a_i a_{i+1} f_i f_{i+1}}{a_i a_{i+1} f_{i+1} (1 + a_{i-1} f_{i-1} + a_{i-1} a_i f_{i-1} f_i)}, \quad r(f_i) = f_i^{-1}, \end{aligned}$$

where  $i \in \mathbb{Z}/3\mathbb{Z}$ . Define  $f$ -functions by

$$f_{0,N}^{n,m} = T_1^n T_2^m T_4^N(f_0), \quad f_{1,N}^{n,m} = T_1^n T_2^m T_4^N(f_1), \quad f_{2,N}^{n,m} = T_1^n T_2^m T_4^N(f_2), \quad (2.9)$$

where  $n, m, N \in \mathbb{Z}$ . These form the edges of a lattice, which we refer to as the  $f$ -lattice, shown in Figure 2. This lattice is three-dimensional, with coordinate axes given by  $n$ ,  $m$ , and  $N$ .

The relations in the  $T_1$ -direction on the lattice:

$$T_1(f_1) = \frac{q c^2}{f_1 f_0} \frac{1 + a_0 f_0}{a_0 + f_0}, \quad T_1(f_0) = \frac{q c^2}{f_0 T_1(f_1)} \frac{1 + a_0 a_2 T_1(f_1)}{a_0 a_2 + T_1(f_1)} \quad (2.10)$$

lead to a system of first-order ordinary difference equations, which is equivalent to  $q$ -P<sub>III</sub> (1.15):

$$f_{1,N}^{n+1,m} = \frac{q^{2N+1} c^2}{f_{1,N}^{n,m} f_{0,N}^{n,m}} \frac{1 + q^n a_0 f_{0,N}^{n,m}}{q^n a_0 + f_{0,N}^{n,m}}, \quad f_{0,N}^{n+1,m} = \frac{q^{2N+1} c^2}{f_{0,N}^{n,m} f_{1,N}^{n+1,m}} \frac{1 + q^{n-m} a_0 a_2 f_{1,N}^{n+1,m}}{q^{n-m} a_0 a_2 + f_{1,N}^{n+1,m}}. \quad (2.11)$$

In a similar manner, in each of the  $T_2$ - and  $T_3$ -directions, we also obtain  $q$ -P<sub>III</sub> (1.15).

In contrast, the action of  $T_4$  on the variables  $f_i$  can be expressed as

$$T_4(f_0) = a_0 a_1 f_1 \frac{1 + a_2 f_2 (a_0 f_0 + 1)}{1 + a_0 f_0 (a_1 f_1 + 1)}, \quad (2.12a)$$

$$T_4(f_1) = a_1 a_2 f_2 \frac{1 + a_0 f_0 (a_1 f_1 + 1)}{1 + a_1 f_1 (a_2 f_2 + 1)}, \quad (2.12b)$$

$$T_4(f_2) = a_2 a_0 f_0 \frac{1 + a_1 f_1 (a_2 f_2 + 1)}{1 + a_2 f_2 (a_0 f_0 + 1)}, \quad (2.12c)$$

or applying  $T_1^n T_2^m T_4^N$  on System (2.12) and using (2.9), we obtain

$$f_{0,N+1}^{n,m} = q^m a_0 a_1 f_{1,N}^{n,m} \frac{1 + q^{-m} a_2 f_{2,N}^{n,m} (q^n a_0 f_{0,N}^{n,m} + 1)}{1 + q^n a_0 f_{0,N}^{n,m} (q^{-n+m} a_1 f_{1,N}^{n,m} + 1)}, \quad (2.13a)$$

$$f_{1,N+1}^{n,m} = q^{-n} a_1 a_2 f_{2,N}^{n,m} \frac{1 + q^n a_0 f_{0,N}^{n,m} (q^{-n+m} a_1 f_{1,N}^{n,m} + 1)}{1 + q^{-n+m} a_1 f_{1,N}^{n,m} (q^{-m} a_2 f_{2,N}^{n,m} + 1)}, \quad (2.13b)$$

$$f_{2,N+1}^{n,m} = q^{n-m} a_2 a_0 f_{0,N}^{n,m} \frac{1 + q^{-n+m} a_1 f_{1,N}^{n,m} (q^{-m} a_2 f_{2,N}^{n,m} + 1)}{1 + q^{-m} a_2 f_{2,N}^{n,m} (q^n a_0 f_{0,N}^{n,m} + 1)}, \quad (2.13c)$$

which is known as a  $q$ -discrete analogue of Painlevé IV equation (denoted by  $q$ -P<sub>IV</sub>) [19].

It is known that discrete dynamical systems of Painlevé type can be also obtained from elements of infinite order of (extended) affine Weyl groups which are not necessarily translations [18]. We introduce the half-translation

$$R_1 = \pi^2 s_1 \quad (2.14)$$

satisfying

$$R_1^2 = T_1. \quad (2.15)$$

Let

$$f_N^M = R_1^M T_4^N(f_0), \quad (2.16)$$

where

$$f_N^{2M-1} = f_{1,N}^{M,0}, \quad f_N^{2M} = f_{0,N}^{M,0}. \quad (2.17)$$

By considering the restricted  $f$ -lattice where  $f_N^M$  are defined (see Figure 3), System (2.11) becomes the following system:

$$f_{1,N}^{n+1,0} = \frac{q^{2N+1} c^2}{f_{1,N}^{n,0} f_{0,N}^{n,0}} \frac{1 + a_0 q^n f_{0,N}^{n,0}}{a_0 q^n + f_{0,N}^{n,0}}, \quad f_{0,N}^{n+1,0} = \frac{q^{2N+1} c^2}{f_{0,N}^{n,0} f_{1,N}^{n+1,0}} \frac{1 + a_2 a_0 q^n f_{1,N}^{n+1,0}}{a_2 a_0 q^n + f_{1,N}^{n+1,0}}, \quad (2.18)$$

which is equivalent to the following single equation:

$$f_N^{M+1} = \frac{q^{2N+1} c^2}{f_N^{M-1} f_N^M} \frac{1 + R_1^M(a_0) f_N^M}{R_1^M(a_0) + f_N^M}. \quad (2.19)$$

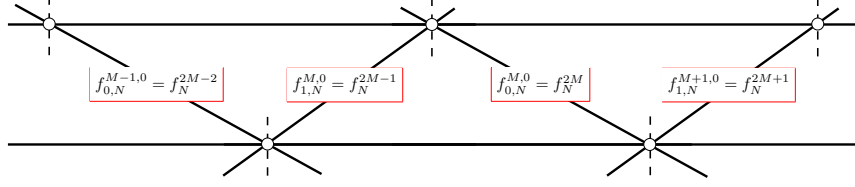
In addition, by assuming  $a_2 = q^{1/2}$ , transformation  $R_1$  becomes the translational motion in the parameter subspace:

$$R_1 : (a_0, a_1) \rightarrow (q^{1/2} a_0, q^{-1/2} a_1), \quad (2.20)$$

then Equation (2.19) can be regarded as the single second-order ordinary difference equation:

$$f_N^{M+1} = \frac{q^{2N+1} c^2}{f_N^{M-1} f_N^M} \frac{1 + a_0 q^{M/2} f_N^M}{a_0 q^{M/2} + f_N^M}, \quad (2.21)$$

which is known as a  $q$ -discrete analogue of Painlevé II equation (denoted by  $q$ -P<sub>II</sub>) [28]. We note that the reduction from System (2.11) to Equation (2.21) is referred to as a symmetrization or a projective reduction [17, 18].

Figure 3. Configuration of  $f$ -functions on the restricted  $f$ -lattice.

**Remark 2.4.** By definition, action of  $\widetilde{W}((A_2 + A_1)^{(1)})$  gives the relations on each edge of the  $f$ -lattice. Since the variables  $f_i$  satisfy Equation (2.8), only two are independent. Therefore, any function associated with an edge on this lattice is determined by two initial edges. This is consistent with the observation that discrete Painlevé equations (which are second order ordinary) are embedded in this lattice.

### 3. QUAD-EQUATIONS OF ABS TYPE FROM THE $\omega$ -LATTICE FOR THE $(A_2 + A_1)^{(1)}$ CASE

In the previous section, we showed how to construct a  $\tau$ -lattice by starting with six initial variables and how to obtain discrete Painlevé equations as relations on the  $f$ -lattice. In this section, we show how to construct a lattice by starting with three variables and applying the action of the extended affine Weyl group to find their iterates. In the resulting  $\omega$ -lattice, we discover higher dimensional integrable partial difference equations, commonly known as quad-equations (because they relate vertices of quadrilaterals), that were classified by Adler *et al.* [1–5].

**3.1. The  $\omega$ -lattice.** Let

$$\kappa_0 = \lambda^{\log a_0 / \log q} k_0, \quad \kappa_1 = \lambda^{\log a_1 / \log q} k_1, \quad \kappa_2 = \lambda^{\log a_2 / \log q} k_2, \quad (3.1)$$

where

$$\lambda = q^{1/2} c. \quad (3.2)$$

Here,  $k_i$  are arbitrary constants satisfying  $k_0 k_1 k_2 = 1$ . The action of  $\widetilde{W}((A_2 + A_1)^{(1)})$  on the parameters  $\kappa_i$  is given by

$$\begin{aligned} s_0 : (\kappa_0, \kappa_1, \kappa_2) &\rightarrow (\kappa_0^{-1}, \kappa_1 \kappa_0, \kappa_2 \kappa_0), \\ s_1 : (\kappa_0, \kappa_1, \kappa_2) &\rightarrow (\kappa_0 \kappa_1, \kappa_1^{-1}, \kappa_2 \kappa_1), \\ s_2 : (\kappa_0, \kappa_1, \kappa_2) &\rightarrow (\kappa_0 \kappa_2, \kappa_1 \kappa_2, \kappa_2^{-1}), \\ \pi : (\kappa_0, \kappa_1, \kappa_2) &\rightarrow (\kappa_1, \kappa_2, \kappa_0), \\ w_0 : (\kappa_0, \kappa_1, \kappa_2) &\rightarrow (a_0 \kappa_0^{-1}, a_1 \kappa_1^{-1}, a_2 \kappa_2^{-1}), \\ w_1 : (\kappa_0, \kappa_1, \kappa_2) &\rightarrow (a_0^{-1} \kappa_0^{-1}, a_1^{-1} \kappa_1^{-1}, a_2^{-1} \kappa_2^{-1}), \\ r : (\kappa_0, \kappa_1, \kappa_2) &\rightarrow (\kappa_0^{-1}, \kappa_1^{-1}, \kappa_2^{-1}). \end{aligned}$$

We note that  $\kappa_i$  satisfy

$$\kappa_0 \kappa_1 \kappa_2 = \lambda. \quad (3.3)$$

From definition (2.3), it follows that the actions of translations  $T_i$ ,  $i = 1, 2, 3, 4$ , on parameters  $\kappa_i$ ,  $i = 0, 1, 2$ , are given by the following:

$$T_1 : (\kappa_0, \kappa_1, \kappa_2) \rightarrow (\lambda \kappa_0, \lambda^{-1} \kappa_1, \kappa_2), \quad (3.4a)$$

$$T_2 : (\kappa_0, \kappa_1, \kappa_2) \rightarrow (\kappa_0, \lambda \kappa_1, \lambda^{-1} \kappa_2), \quad (3.4b)$$

$$T_3 : (\kappa_0, \kappa_1, \kappa_2) \rightarrow (\lambda^{-1} \kappa_0, \kappa_1, \lambda \kappa_2), \quad (3.4c)$$

$$T_4 : (\kappa_0, \kappa_1, \kappa_2) \rightarrow (a_0 \kappa_0, a_1 \kappa_1, a_2 \kappa_2). \quad (3.4d)$$



Now we are in a position to define the three initial variables

$$\omega_0 = \frac{\kappa_2^{1/3}}{\kappa_1^{1/3}} \frac{\bar{\tau}_0}{\tau_0}, \quad \omega_1 = \frac{\kappa_0^{1/3}}{\kappa_2^{1/3}} \frac{\bar{\tau}_1}{\tau_1}, \quad \omega_2 = \frac{\kappa_1^{1/3}}{\kappa_0^{1/3}} \frac{\bar{\tau}_2}{\tau_2}, \quad (3.5)$$

whose iterates (constructed below) will provide us with the  $\omega$ -lattice.

The action of  $\widetilde{W}((A_2 + A_1)^{(1)})$  on these variables  $\omega_i$  is given by the following lemma, which follows from the above definitions.

**Lemma 3.1.** *The action of  $\widetilde{W}((A_2 + A_1)^{(1)})$  on variables  $\omega_i$  is given by*

$$\begin{aligned} s_i(\omega_i) &= \omega_i \frac{a_i \lambda \omega_{i+1} + \kappa_i \omega_{i+2}}{\lambda \omega_{i+1} + a_i \kappa_i \omega_{i+2}}, & s_i(\omega_{i+1}) &= \kappa_i^{-1} \omega_{i+1}, & s_i(\omega_{i+2}) &= \kappa_i \omega_{i+2}, \\ \pi(\omega_i) &= \omega_{i+1}, & w_0(\omega_i) &= \frac{a_{i+1} \kappa_i \kappa_{i+1} \omega_i + a_{i+1} a_{i+2} \omega_{i+1} + \kappa_i \lambda \omega_{i+2}}{a_{i+1} \kappa_i \kappa_{i+2} \omega_{i+1} \omega_{i+2}}, \\ w_1(\omega_i) &= \frac{a_{i+1} \kappa_i \kappa_{i+1} \omega_i}{a_{i+1} \kappa_i \kappa_{i+2} \omega_{i+1} \omega_{i+2} + a_{i+1} a_{i+2} \kappa_i \lambda \omega_i \omega_{i+2} + \omega_i \omega_{i+1}}, & r(\omega_i) &= \omega_i^{-1}, \end{aligned}$$

where  $i \in \mathbb{Z}/3\mathbb{Z}$ .

We define  $\omega$ -functions by

$$\omega_{l_1, l_2, l_3, l_4} = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4}(\omega_0), \quad (3.6)$$

where  $l_1, l_2, l_3, l_4 \in \mathbb{Z}$  and the  $\omega$ -lattice is as shown in Figure 4. We note that

$$\omega_0 = \omega_{0,0,0,0}, \quad \omega_1 = \kappa_2^{-1} \omega_{1,0,0,0}, \quad \omega_2 = \kappa_1 \omega_{1,1,0,0}. \quad (3.7)$$

**Lemma 3.2.** *Since for all  $w \in \widetilde{W}((A_2 + A_1)^{(1)})$ ,*

$$w(\omega_i) \in \mathcal{L} \quad (i = 0, 1, 2), \quad (3.8)$$

where  $\mathcal{L} = \mathcal{K}(\omega_0, \omega_1, \omega_2)$  is the field of rational functions in  $\omega_i$ ,  $i = 0, 1, 2$ , with coefficient field  $\mathcal{K} = \mathbb{C}(a_i, \kappa_i, \lambda)$ , every point on the  $\omega$ -lattice is determined by three initial points. This implies that quad-equations appear as relations on the  $\omega$ -lattice. Moreover, relations on the  $f$ -lattice can be expressed by those on the  $\omega$ -lattice because of the following correspondence:

$$f_0 = \kappa_1 \kappa_2 \frac{\omega_1}{\omega_2}, \quad \left( \text{or } f_{0, l_4}^{l_1 - l_3, l_2 - l_3} = \frac{\omega_{l_1+1, l_2, l_3, l_4}}{\omega_{l_1+1, l_2+1, l_3, l_4}} \right), \quad (3.9a)$$

$$f_1 = \kappa_2 \kappa_0 \frac{\omega_2}{\omega_0}, \quad \left( \text{or } f_{1, l_4}^{l_1 - l_3, l_2 - l_3} = q^{l_4} \lambda \frac{\omega_{l_1+1, l_2+1, l_3, l_4}}{\omega_{l_1, l_2, l_3, l_4}} \right), \quad (3.9b)$$

$$f_2 = \kappa_0 \kappa_1 \frac{\omega_0}{\omega_1}, \quad \left( \text{or } f_{2, l_4}^{l_1 - l_3, l_2 - l_3} = q^{l_4} \lambda \frac{\omega_{l_1, l_2, l_3, l_4}}{\omega_{l_1+1, l_2, l_3, l_4}} \right). \quad (3.9c)$$

We have constructed the  $\omega$ -lattice associated with  $\widetilde{W}((A_2 + A_1)^{(1)})$ . Henceforth, let us consider the quad-equations appearing on the  $\omega$ -lattice.

**Lemma 3.3.** *The following quad-equations:*

$$\frac{\omega_{l_1+1, l_2, l_3+1, l_4}}{\omega_{l_1, l_2, l_3, l_4}} = \frac{q^{l_1 - l_3 - 1} a_0 \omega_{l_1+1, l_2, l_3, l_4} - \omega_{l_1, l_2, l_3+1, l_4}}{q^{l_1 - l_3 - 1} a_0 \omega_{l_1, l_2, l_3+1, l_4} - \omega_{l_1+1, l_2, l_3, l_4}}, \quad (3.10a)$$

$$\frac{\omega_{l_1+1, l_2+1, l_3, l_4}}{\omega_{l_1, l_2, l_3, l_4}} = \frac{q^{-l_1 + l_2 + l_4} \lambda a_1 \omega_{l_1, l_2+1, l_3, l_4} - \omega_{l_1+1, l_2, l_3, l_4}}{q^{l_4} \lambda (q^{-l_1 + l_2} a_1 \omega_{l_1+1, l_2, l_3, l_4} - q^{l_4} \lambda \omega_{l_1, l_2+1, l_3, l_4})}, \quad (3.10b)$$

$$\frac{\omega_{l_1, l_2+1, l_3+1, l_4}}{\omega_{l_1, l_2, l_3, l_4}} = \frac{q^{-l_2 + l_3} a_2 \omega_{l_1, l_2, l_3+1, l_4} - q^{l_4} \lambda \omega_{l_1, l_2+1, l_3, l_4}}{q^{l_4} \lambda (q^{-l_2 + l_3 + l_4} a_2 \lambda \omega_{l_1, l_2+1, l_3, l_4} - \omega_{l_1, l_2, l_3+1, l_4})}, \quad (3.10c)$$

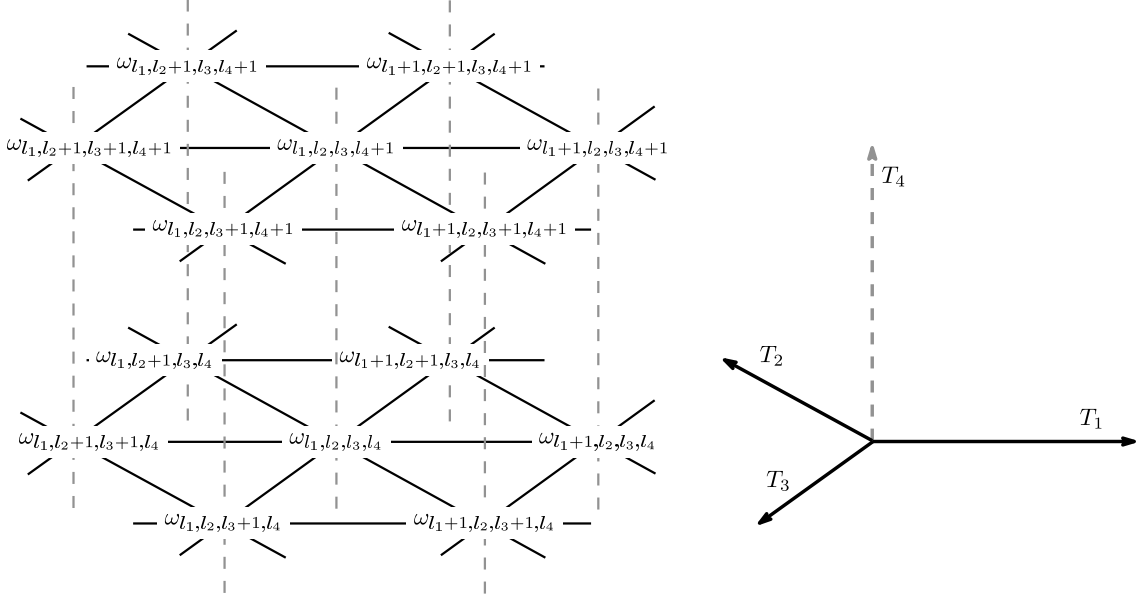


Figure 4. Configuration of  $\omega$ -functions on the  $\omega$ -lattice. Note that each  $\omega$ -function is defined on the intersection of four lines.

$$\frac{\omega_{l_1+1, l_2, l_3, l_4+1}}{\omega_{l_1, l_2, l_3, l_4}} - \frac{\omega_{l_1, l_2, l_3, l_4+1}}{\omega_{l_1+1, l_2, l_3, l_4}} = \frac{q^{2l_4+1}\lambda^2 - 1}{q^{-l_1+l_2+l_4}\lambda a_1}, \quad (3.11a)$$

$$\frac{\omega_{l_1, l_2+1, l_3, l_4+1}}{\omega_{l_1, l_2, l_3, l_4}} - \frac{1}{q^{2l_4+1}\lambda^2} \frac{\omega_{l_1, l_2, l_3, l_4+1}}{\omega_{l_1, l_2+1, l_3, l_4}} = \frac{q^{2l_4+1}\lambda^2 - 1}{q^{2l_4+1}\lambda^2}, \quad (3.11b)$$

$$\frac{\omega_{l_1, l_2, l_3+1, l_4+1}}{\omega_{l_1, l_2, l_3, l_4}} - \frac{\omega_{l_1, l_2, l_3, l_4+1}}{\omega_{l_1, l_2, l_3+1, l_4}} = \frac{a_2(q^{2l_4+1}\lambda^2 - 1)}{q^{l_2-l_3+l_4}\lambda}, \quad (3.11c)$$

and an additional partial difference equation:

$$\begin{aligned} \omega_{l_1, l_2, l_3, l_4+1} = & \frac{(q^{2(-l_1+l_2)}a_1^2 - 1)\omega_{l_1+1, l_2, l_3, l_4}\omega_{l_1, l_2+1, l_3, l_4}}{q^{-l_1+l_2}a_1(q^{-l_1+l_2}a_1\omega_{l_1+1, l_2, l_3, l_4} - q^{l_4}\lambda\omega_{l_1, l_2+1, l_3, l_4})} \\ & + \frac{q^{-l_2+l_3}a_2(q^{-l_1+l_2+l_4}\lambda a_1\omega_{l_1, l_2+1, l_3, l_4} - \omega_{l_1+1, l_2, l_3, l_4})\omega_{l_1, l_2, l_3, l_4}}{q^{-l_1+l_2}a_1\omega_{l_1+1, l_2, l_3, l_4} - q^{l_4}\lambda\omega_{l_1, l_2+1, l_3, l_4}}, \end{aligned} \quad (3.12)$$

hold on the  $\omega$ -lattice. We note that Equations (3.10) are  $H3_{(\delta, \epsilon)=(0,0)}$ -type equations, Equations (3.11) are  $D4_{(\delta_1, \delta_2, \delta_3)=(1,0,0)}$ -type equations and Equation (3.12) is a  $H3_{(\delta, \epsilon)=(0,1)}$ -type equation.

*Proof.* First, we prove System (3.10). From the action of translations  $T_1$ ,  $T_2$  and  $T_3$ , it holds that

$$\frac{\omega_1}{\omega_2} = \frac{a_0\kappa_2\kappa_0T_1(\omega_2) - \omega_0}{a_0\kappa_2\kappa_1\omega_0 - \lambda\kappa_2T_1(\omega_2)}, \quad (3.13a)$$

$$\frac{\omega_2}{\omega_0} = \frac{a_1\kappa_0\kappa_1T_2(\omega_0) - \omega_1}{a_1\kappa_0\kappa_2\omega_1 - \lambda\kappa_0T_2(\omega_0)}, \quad (3.13b)$$

$$\frac{\omega_0}{\omega_1} = \frac{a_2\kappa_1\kappa_2T_3(\omega_1) - \omega_2}{a_2\kappa_1\kappa_0\omega_2 - \lambda\kappa_1T_3(\omega_1)}. \quad (3.13c)$$

Applying  $T_1^{l_1} T_2^{l_2} T_3^{l_3+1} T_4^{l_4}$ ,  $T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4}$  and  $T_1^{l_1} T_2^{l_2+1} T_3^{l_3+1} T_4^{l_4}$  on Equations (3.13a)–(3.13c), we obtain Equations (3.10a)–(3.10c), respectively.

We now consider the derivation of System (3.11). From the action of the translation  $T_4$ , it easily verified that

$$a_2 \kappa_2 \frac{T_4(\omega_1)}{\omega_0} - \frac{1}{\kappa_2} \frac{T_4(\omega_0)}{\omega_1} = \frac{q\lambda^2 - 1}{\lambda a_1}, \quad (3.14a)$$

$$\frac{1}{a_1 \kappa_1 \kappa_2} \frac{T_4(\omega_2)}{\omega_1} - \frac{a_2 \kappa_1 \kappa_2}{q\lambda^2} \frac{T_4(\omega_1)}{\omega_2} = \frac{q\lambda^2 - 1}{q\lambda^2}, \quad (3.14b)$$

$$\kappa_1 \frac{T_4(\omega_0)}{\omega_2} - \frac{1}{a_1 \kappa_1} \frac{T_4(\omega_2)}{\omega_0} = \frac{a_2(q\lambda^2 - 1)}{q\lambda}. \quad (3.14c)$$

Applying  $T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4}$ ,  $T_1^{l_1-1} T_2^{l_2} T_3^{l_3} T_4^{l_4}$  and  $T_1^{l_1-1} T_2^{l_2-1} T_3^{l_3} T_4^{l_4}$  on Equations (3.14a)–(3.14c), we obtain Equations (3.11a)–(3.11c), respectively.

Finally, we prove Equation (3.12). By eliminating  $\omega_2$  from Equation (3.13b) and

$$T_4(\omega_0) = \frac{\omega_0 \omega_1 + a_1 \kappa_0 (\lambda a_2 \omega_0 + \kappa_2 \omega_1) \omega_2}{a_1 \kappa_0 \kappa_1 \omega_0}, \quad (3.15)$$

the following relation can be derived:

$$T_4(\omega_0) = \frac{(a_1^2 - 1) \omega_1 T_2(\omega_0) + a_1 a_2 (a_1 \kappa_0 \kappa_1 T_2(\omega_0) - \omega_1) \omega_0}{a_1 (a_1 \omega_1 - \kappa_0 \kappa_1 T_2(\omega_0))}. \quad (3.16)$$

Applying  $T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4}$  on Equation (3.16), we obtain Equation (3.12). This completes the proof.  $\square$

In [13, 14], we showed the following proposition by using Lemma 3.3:

**Proposition 3.4** ([13, 14]). *The  $\omega$ -lattice can be obtained from an asymmetric 4D cube which has twelve  $H3_{(\delta, \epsilon)=(0,0)}$ -type equations and twelve  $D4_{(\delta_1, \delta_2, \delta_3)=(1,0,0)}$ -type equations associated with each face.*

It follows from Proposition 3.4 that the above quad-equations are the only ones that relate four points on the  $\omega$ -variables. Therefore, we have shown a part of Theorem 1.1.

**3.2. Relations to discrete Schwarzian KdV equation.** In this section, we show how to obtain the discrete Schwarzian KdV equation, which is known as a discrete analogue of the Cauchy-Riemann relation (cross-ratio equation), from the  $\omega$ -lattice.

In a recent work [11], Hay *et al.* showed that by setting

$$z_{l_1, l_2, l_3, l_4} = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4}(z), \quad (3.17)$$

where

$$z = e^{2 \log(a_1^{-1} a_2) / 3 \log q} \frac{a_2^{2/3}}{a_1^{2/3}} \frac{T_4(\bar{\tau}_0)}{\tau_0}, \quad (3.18)$$

one can obtain the discrete Schwarzian KdV equation:

$$\frac{(z_{l_1, l_2, l_3, l_4} - z_{l_1+1, l_2, l_3, l_4})(z_{l_1, l_2, l_3+1, l_4} - z_{l_1+1, l_2, l_3+1, l_4})}{(z_{l_1, l_2, l_3, l_4} - z_{l_1, l_2, l_3+1, l_4})(z_{l_1+1, l_2, l_3, l_4} - z_{l_1+1, l_2, l_3+1, l_4})} = q^{2(l_1-l_3-1)} a_0^2. \quad (3.19)$$

Equation (3.19) can be found from the  $\omega$ -lattice because of the following relation:

$$z = T_4(\omega_0) \omega_0, \quad (\text{or } z_{l_1, l_2, l_3, l_4} = \omega_{l_1, l_2, l_3, l_4+1} \omega_{l_1, l_2, l_3, l_4}). \quad (3.20)$$

Furthermore, we can also obtain the following equations:

$$\frac{(z_{l_1, l_2, l_3, l_4} - q^{2l_4+1} \lambda^2 z_{l_1, l_2+1, l_3, l_4})(z_{l_1+1, l_2, l_3, l_4} - q^{2l_4+1} \lambda^2 z_{l_1+1, l_2+1, l_3, l_4})}{(z_{l_1, l_2, l_3, l_4} - z_{l_1+1, l_2, l_3, l_4})(z_{l_1, l_2+1, l_3, l_4} - z_{l_1+1, l_2+1, l_3, l_4})} = q^{-2l_1+2l_2+2l_4+1} \lambda^2 a_1^2, \quad (3.21a)$$

$$\frac{(z_{l_1, l_2, l_3, l_4} - z_{l_1, l_2, l_3+1, l_4})(z_{l_1, l_2+1, l_3, l_4} - z_{l_1, l_2+1, l_3+1, l_4})}{(z_{l_1, l_2, l_3, l_4} - q^{2l_4+1} \lambda^2 z_{l_1, l_2+1, l_3, l_4})(z_{l_1, l_2, l_3+1, l_4} - q^{2l_4+1} \lambda^2 z_{l_1, l_2+1, l_3+1, l_4})} = q^{-2l_2+2l_3-2l_4-1} \lambda^{-2} a_2^2, \quad (3.21b)$$

$$\frac{(z_{l_1, l_2, l_3, l_4} - z_{l_1+1, l_2, l_3, l_4})(z_{l_1, l_2, l_3, l_4+1} - z_{l_1+1, l_2, l_3, l_4+1})}{z_{l_1, l_2, l_3, l_4} z_{l_1+1, l_2, l_3, l_4}} = \frac{(q^{2l_4+1} \lambda^2 - 1)(q^{2l_4+3} \lambda^2 - 1)}{q^{2(-l_1+l_2+l_4)+1} \lambda^2 a_1^2}, \quad (3.21c)$$

$$\frac{(z_{l_1, l_2, l_3, l_4} - q^{2l_4+1} \lambda^2 z_{l_1, l_2+1, l_3, l_4})(z_{l_1, l_2, l_3, l_4+1} - q^{2l_4+3} \lambda^2 z_{l_1, l_2+1, l_3, l_4+1})}{z_{l_1, l_2, l_3, l_4} z_{l_1, l_2+1, l_3, l_4}} = (q^{2l_4+1} \lambda^2 - 1)(q^{2l_4+3} \lambda^2 - 1), \quad (3.21d)$$

$$\frac{(z_{l_1, l_2, l_3, l_4} - z_{l_1, l_2, l_3+1, l_4})(z_{l_1, l_2, l_3, l_4+1} - z_{l_1, l_2, l_3+1, l_4+1})}{z_{l_1, l_2, l_3, l_4} z_{l_1, l_2, l_3+1, l_4}} = \frac{(q^{2l_4+1} \lambda^2 - 1)(q^{2l_4+3} \lambda^2 - 1)}{q^{2(l_2-l_3+l_4)+1} \lambda^2 a_2^{-2}}, \quad (3.21e)$$

from the following relations:

$$\frac{(z - q\lambda^2 T_2(z))(T_1(z) - q\lambda^2 T_1 T_2(z))}{(z - T_1(z))(T_2(z) - T_1 T_2(z))} = q\lambda^2 a_1^2, \quad (3.22a)$$

$$\frac{(z - T_3(z))(T_2(z) - T_2 T_3(z))}{(z - q\lambda^2 T_2(z))(T_3(z) - q\lambda^2 T_2 T_3(z))} = q^{-1} \lambda^{-2} a_2^2, \quad (3.22b)$$

$$\frac{(z - T_1(z))(T_4(z) - T_1 T_4(z))}{z T_1(z)} = \frac{(q\lambda^2 - 1)(q^3 \lambda^2 - 1)}{q\lambda^2 a_1^2}, \quad (3.22c)$$

$$\frac{(z - q\lambda^2 T_2(z))(T_4(z) - q^3 \lambda^2 T_2 T_4(z))}{z T_2(z)} = (q\lambda^2 - 1)(q^3 \lambda^2 - 1), \quad (3.22d)$$

$$\frac{(z - T_3(z))(T_4(z) - T_3 T_4(z))}{z T_3(z)} = \frac{(q\lambda^2 - 1)(q^3 \lambda^2 - 1)}{q\lambda^2 a_2^{-2}}. \quad (3.22e)$$

We note that Equations (3.19), (3.21a) and (3.21b) are  $Q1_{\epsilon=0}$ -type equations, while Equations (3.21c)–(3.21e) are  $H1_{\epsilon=0}$ -type equations.

**3.3. The restricted  $\omega$ -lattice.** In the case of the  $f$ -lattice, we showed that System (2.11) can be rewritten as the single equation (2.21) (or (2.19)) on the restricted lattice (projective reduction). The concept of projective reduction applies not only for the  $f$ -lattice but also for the  $\omega$ -lattice. Let

$$\omega_{l, l_4} = R_1^{l_4} T_4^{l_4}(\omega_0), \quad (3.23)$$

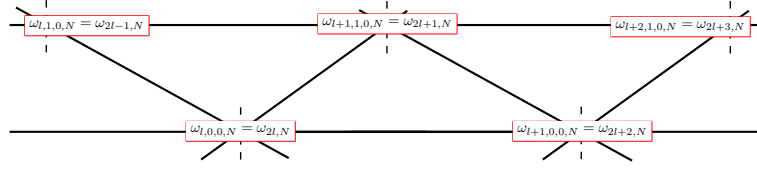
where

$$\omega_{2l-1, l_4} = \frac{q^{l_4} \lambda}{a_2^{l_4} \kappa_2} \omega_{l, 1, 0, l_4}, \quad \omega_{2l, l_4} = \omega_{l, 0, 0, l_4}. \quad (3.24)$$

Considering the restricted  $\omega$ -lattice where  $\omega_{l, l_4}$  are defined (see Figure 5), Equations (3.10a) and (3.10b) can be rewritten as

$$\frac{\omega_{l+2, 1, 0, l_4}}{\omega_{l, 0, 0, l_4}} = \frac{\omega_{l+1, 1, 0, l_4} + q^{l_4} a_0 \omega_{l+1, 0, 0, l_4}}{\omega_{l+1, 0, 0, l_4} + q^{l_4} a_0 \omega_{l+1, 1, 0, l_4}}, \quad (3.25a)$$

$$\frac{\omega_{l+1, 0, 0, l_4}}{\omega_{l, 1, 0, l_4}} = \frac{q^{l_4} \lambda (q^{-l_4} a_1 \omega_{l, 0, 0, l_4} + q^{l_4} \lambda \omega_{l+1, 1, 0, l_4})}{q^{-l+l_4} \lambda a_1 \omega_{l+1, 1, 0, l_4} + \omega_{l, 0, 0, l_4}}, \quad (3.25b)$$

Figure 5. Configuration of  $\omega$ -functions on the restricted  $\omega$ -lattice.

respectively. The system of equations (3.25a) and (3.25b) is expressed by the single equation

$$\frac{\omega_{l+3,l_4}}{\omega_{l,l_4}} = \frac{q^{l_4} \lambda (R_1^l T_4^{l_4}(\kappa_2) \omega_{l+1,l_4} + q^{l_4} \lambda R_1^l(a_0) \omega_{l+2,l_4})}{R_1^l T_4^{l_4}(\kappa_2) (q^{l_4} \lambda \omega_{l+2,l_4} + R_1^l(a_0) R_1^l T_4^{l_4}(\kappa_2) \omega_{l+1,l_4})}. \quad (3.26)$$

In a similar manner, Equation (3.11a) becomes

$$\frac{\omega_{l+2,l_4+1}}{\omega_{l,l_4}} - \frac{\omega_{l,l_4+1}}{\omega_{l+2,l_4}} = \frac{q^{2l_4+1} \lambda^2 - 1}{q^{l_4} \lambda R_1^l(a_1)}, \quad (3.27)$$

and furthermore, Equations (3.11b) and (3.11c) can be expressed by

$$\frac{\omega_{l,l_4+1}}{\omega_{l+1,l_4}} - \frac{1}{R_1^{l-1} T_4^{l_4}(a_2 \kappa_2^2)} \frac{\omega_{l+1,l_4+1}}{\omega_{l,l_4}} = \frac{q^{2l_4+1} \lambda^2 - 1}{q^{l_4} \lambda R_1^{l-1} T_4^{l_4}(a_2 \kappa_2^2)}, \quad (3.28)$$

on the restricted  $\omega$ -lattice.

**Remark 3.5** ([13]). *The restricted  $\omega$ -lattice can be also obtained from the asymmetric 4D cube.*

We note that on the restricted  $\omega$ -lattice, Equations (3.19) and (3.21a), Equation (3.21c) and Equations (3.21d) and (3.21e) can be also rewritten as

$$\frac{(\zeta_{l,l_4} - \zeta_{l+2,l_4})(\zeta_{l+1,l_4} - \zeta_{l+3,l_4})}{(q^{l_4+1/2} \lambda \zeta_{l,l_4} - \zeta_{l+1,l_4})(q^{l_4+1/2} \lambda \zeta_{l+2,l_4} - \zeta_{l+3,l_4})} = q^{-l_4-1/2} \lambda^{-1} R_1^l(a_0^2), \quad (3.29a)$$

$$\frac{(\zeta_{l,l_4} - \zeta_{l+2,l_4})(\zeta_{l+1,l_4+1} - \zeta_{l+2,l_4+1})}{\zeta_{l,l_4} \zeta_{l+2,l_4}} = \frac{(q^{2l_4+1} \lambda^2 - 1)(q^{2l_4+3} \lambda^2 - 1)}{q^{2l_4+1} \lambda^2 R_1^l(a_1^2 a_2)}, \quad (3.29b)$$

$$\frac{(\zeta_{l,l_4} - q^{-(l_4+1)/2} \lambda^{-1} \zeta_{l+1,l_4})(\zeta_{l+1,l_4+1} - q^{-(l_4+3)/2} \lambda^{-1} \zeta_{l+1,l_4+1})}{\zeta_{l,l_4} \zeta_{l+1,l_4}} = \frac{(q^{2l_4+1} \lambda^2 - 1)(q^{2l_4+3} \lambda^2 - 1)}{q^{3l_4+5/2} \lambda^3 R_1^l(a_0 a_1)}, \quad (3.29c)$$

where

$$\zeta_{l,l_4} = R_1^l T_4^{l_4}(a_2^{-1/2} \kappa_2^{-1}) R_1^l T_4^{l_4}(z) = R_1^l T_4^{l_4}(a_2^{-1/2} \kappa_2^{-1}) \omega_{l,l_4} \omega_{l,l_4+1}. \quad (3.30)$$

From periodic reduction of a partial difference equation, we can obtain a quad-equation on the restricted  $\omega$ -lattice. In fact, we obtain the following lemmas.

**Lemma 3.6.** *Equation (3.26) is equivalent to the  $(1, -2)$ -periodic reduction of  $H3_{(\delta, \epsilon)=(0,0)}$  (1.5).*

*Proof.* By setting

$$\Omega_{l_1}^{l_4} = R_1^l T_4^{l_4}(\lambda^{2/3} \kappa_1) \omega_{l,l_4}, \quad (3.31)$$

Equation (3.26) can be rewritten as

$$\frac{\Omega_{l+3}^{l_4}}{\Omega_l^{l_4}} = \frac{\Omega_{l+1}^{l_4} + q^{l_4} \lambda R_1^l(a_0) \Omega_{l+2}^{l_4}}{q^{l_4} \lambda (q^{l_4} \lambda \Omega_{l+2}^{l_4} + R_1^l(a_0) \Omega_{l+1}^{l_4})}, \quad (3.32)$$

which is equivalent to Equation (1.12) with the following correspondence:

$$\Omega_{0,0} = \Omega_0^0, \quad \frac{\alpha_0}{\beta_0} = a_0, \quad \bar{\phantom{x}} = T_1, \quad \hat{\phantom{x}} = R_1. \quad (3.33)$$

This completes the proof.  $\square$

**Lemma 3.7.** Equation (3.29a) can be obtained by a periodic reduction of  $Q1_{\epsilon=0}$  (1.4) and the reduction is defined by

$$U_{l,m} = q^{-m/2} \lambda^{-m} \zeta_l^m, \quad (3.34)$$

with the  $(1, -2)$ -periodic condition

$$\zeta_{l+1}^{m-2} = \zeta_l^m. \quad (3.35)$$

*Proof.* The periodic condition (3.35) implies the condition on the parameters

$$\frac{\bar{\alpha}}{\alpha} = \frac{\beta}{\widehat{\widehat{\beta}}} = q^2. \quad (3.36)$$

Therefore, Equation (1.4) can be reduced to

$$\frac{(\zeta - \widehat{\widehat{\zeta}})(\widehat{\widehat{\zeta}} - \widehat{\widehat{\zeta}})}{(q^{1/2} \lambda \zeta - \widehat{\widehat{\zeta}})(q^{1/2} \lambda \widehat{\widehat{\zeta}} - \widehat{\widehat{\zeta}})} = \frac{\alpha}{q^{1/2} \lambda \beta}, \quad (3.37)$$

where

$$\zeta = \zeta_l^m, \quad \alpha = \alpha_l, \quad \beta = \beta_m. \quad (3.38)$$

Then, the statement holds since Equation (3.29a) is equivalent to Equation (3.37) with the following correspondence:

$$\zeta_0^0 = \zeta_{0,0}, \quad \frac{\alpha_0}{\beta_0} = a_0^2, \quad \bar{\phantom{x}} = T_1, \quad \hat{\phantom{x}} = R_1. \quad (3.39)$$

$\square$

#### 4. $\omega$ -LATTICE FOR THE $(A_1 + A'_1)^{(1)}$ CASE

In this section, we construct the  $\omega$ -lattice associated with the extended affine Weyl group of type  $(A_1 + A_1)^{(1)}$ .

**4.1. The  $\tau$ -lattice,  $f$ -lattice and  $\omega$ -lattice.** In this section, we first construct the  $\tau$  functions of  $A_6^{(1)}$ -surface  $q$ -Painlevé systems with the extended affine Weyl group of type  $(A_1 + A_1)^{(1)}$ . Then, we construct the  $\tau$ -lattice, discrete Painlevé lattice ( $f$ -lattice) and  $\omega$ -lattice. The actions (4.1) were first obtained by Yamada [35].

The transformation group  $\widetilde{W}((A_1 + A'_1)^{(1)})$  has 5 generators  $s_0, s_1, w_0, w_1, \pi$ . Below, we describe their actions on parameters:  $a_0, a_1, b$ , and on variables:  $\tau_i, i = -3, \dots, 1$ .

**Lemma 4.1.** The action of  $\widetilde{W}((A_1 + A'_1)^{(1)})$  on parameters are given by

$$\begin{aligned} s_0 : (a_0, a_1, b) &\mapsto \left( \frac{1}{a_0}, a_0^2 a_1, \frac{b}{a_0} \right), & s_1 : (a_0, a_1, b) &\mapsto \left( a_0 a_1^2, \frac{1}{a_1}, a_1 b \right), \\ w_0 : (a_0, a_1, b) &\mapsto \left( \frac{1}{a_0}, \frac{1}{a_1}, \frac{b}{a_0} \right), & w_1 : (a_0, a_1, b) &\mapsto \left( \frac{1}{a_0}, \frac{1}{a_1}, \frac{b}{a_0^2 a_1} \right), \\ \pi : (a_0, a_1, b) &\mapsto \left( \frac{1}{a_1}, \frac{1}{a_0}, \frac{b}{a_0 a_1} \right), \end{aligned}$$

while its actions on variables are given by

$$s_0 : (\tau_{-3}, \tau_{-1}, \tau_1) \mapsto \left( \frac{a_0 \tau_1 \tau_{-2}^2 + \tau_{-1} \tau_0 \tau_{-2} + \tau_{-3} \tau_0^2}{a_0 \tau_{-1} \tau_1}, \frac{a_0 \tau_0^2 + b \tau_{-2} \tau_2}{a_0 \tau_1}, \frac{b \tau_{-2} \tau_2 + \tau_0^2}{\tau_{-1}} \right), \quad (4.1a)$$

$$s_1 : (\tau_{-2}, \tau_0) \mapsto \left( \frac{a_0 a_1 \tau_{-1}^2 + b \tau_{-3} \tau_1}{a_0 a_1 \tau_0}, \frac{a_0 \tau_{-1}^2 + b \tau_{-3} \tau_1}{a_0 \tau_{-2}} \right), \quad (4.1b)$$

$$w_0 : (\tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_1) \mapsto (\tau_3, \tau_2, \tau_1, \tau_{-1}), \quad (4.1c)$$

$$w_1 : (\tau_{-3}, \tau_{-2}, \tau_0, \tau_1) \mapsto (\tau_1, \tau_0, \tau_{-2}, \tau_{-3}), \quad (4.1d)$$

$$\pi : (\tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1) \mapsto (\tau_2, \tau_1, \tau_0, \tau_{-1}, \tau_{-2}), \quad (4.1e)$$

where

$$\tau_2 = \frac{a_0 (\tau_{-1} \tau_0 + \tau_{-2} \tau_1)}{b \tau_{-3}}, \quad \tau_3 = \frac{\tau_0 \tau_1 + \tau_{-1} \tau_2}{b \tau_{-2}}. \quad (4.2)$$

For each element  $w \in \widetilde{W}((A_1 + A'_1)^{(1)})$  and function  $F = F(a_i, b, \tau_j)$ , we use the notation  $w.F$  to mean  $w.F = F(w.a_i, w.b, w.\tau_j)$ , that is,  $w$  acts on the arguments from the left.

The proof of Lemma 4.1 is given in Appendix A. We note that the group of transformations  $\widetilde{W}((A_1 + A'_1)^{(1)}) = \langle s_0, s_1, w_0, w_1, \pi \rangle$  forms the extended affine Weyl group of type  $(A_1 + A_1)^{(1)}$ . Namely, the transformations satisfy the fundamental relations

$$s_0^2 = s_1^2 = (s_0 s_1)^\infty = 1, \quad w_0^2 = w_1^2 = (w_0 w_1)^\infty = 1, \quad (4.3a)$$

$$\pi^2 = 1, \quad \pi s_0 = s_1 \pi, \quad \pi w_0 = w_1 \pi, \quad (4.3b)$$

and the action of  $W(A_1^{(1)}) = \langle s_0, s_1 \rangle$  and that of  $W(A_{1, |\beta|^2=14}^{(1)}) = \langle w_0, w_1 \rangle$  commute. We note that the relation  $(ww')^\infty = 1$  for transformations  $w$  and  $w'$  means that there is no positive integer  $N$  such that  $(ww')^N = 1$ .

To iterate each variable  $\tau_i$ , we need the following translations  $T_i$ ,  $i = 1, 2, 3$ , defined by

$$T_1 = w_0 w_1, \quad T_2 = \pi s_1 w_0, \quad T_3 = \pi s_0 w_0. \quad (4.4)$$

Note that  $T_i$ ,  $i = 1, 2, 3$ , commute with each other and  $T_1 T_2 T_3 = 1$ . The actions of these on the parameters are given by

$$T_1 : (a_0, a_1, b) \rightarrow (a_0, a_1, qb), \quad (4.5a)$$

$$T_2 : (a_0, a_1, b) \rightarrow (qa_0, q^{-1}a_1, b), \quad (4.5b)$$

$$T_3 : (a_0, a_1, b) \rightarrow (q^{-1}a_0, qa_1, q^{-1}b), \quad (4.5c)$$

where the parameter

$$q = a_0 a_1 \quad (4.6)$$

is invariant under the action of translations. We define  $\tau$  functions by

$$\tau_N^n = T_1^n T_2^N (\tau_{-3}), \quad (4.7)$$

where  $n, N \in \mathbb{Z}$  and the  $\tau$ -lattice is as shown in Figure 6. We note that

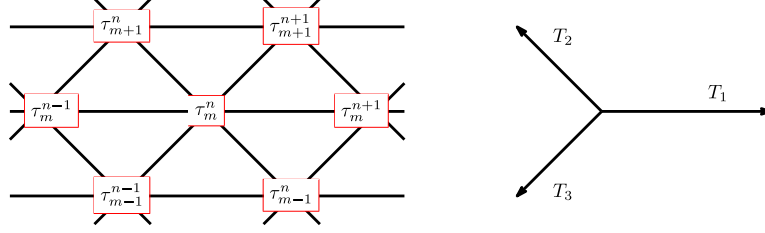
$$\tau_{-3} = \tau_0^0, \quad \tau_{-2} = \tau_1^1, \quad \tau_{-1} = \tau_0^1, \quad \tau_0 = \tau_1^2, \quad \tau_1 = \tau_0^2, \quad \tau_2 = \tau_1^3, \quad \tau_3 = \tau_0^3. \quad (4.8)$$

Next, we consider the  $f$ -lattice. Let

$$f_0 = \frac{\tau_{-2} \tau_1}{\tau_{-1} \tau_0}, \quad f_1 = \frac{\tau_{-3} \tau_0}{\tau_{-2} \tau_{-1}}, \quad f_2 = \frac{\tau_{-1}^2}{\tau_{-3} \tau_1}, \quad (4.9)$$

where

$$f_0 f_1 f_2 = 1. \quad (4.10)$$

Figure 6. Configuration of  $\tau$  functions on the  $\tau$ -lattice.

The action of  $\widetilde{W}((A_1 + A'_1)^{(1)})$  on the variables  $f_i$ ,  $i = 0, 1, 2$ , is given by

$$\begin{aligned} s_0 : (f_0, f_1, f_2) &\mapsto \left( \frac{f_0(a_0 f_0 + a_0 + f_1)}{f_0 + f_1 + 1}, \frac{f_1(a_0 f_0 + f_1 + 1)}{a_0(f_0 + f_1 + 1)}, \frac{a_0 f_2(f_0 + f_1 + 1)^2}{(a_0 f_0 + a_0 + f_1)(a_0 f_0 + f_1 + 1)} \right), \\ s_1 : (f_0, f_1) &\mapsto \left( \frac{f_0(a_0 a_1 + b f_0 f_1)}{a_1(a_0 + b f_0 f_1)}, \frac{a_1 f_1(a_0 + b f_0 f_1)}{a_0 a_1 + b f_0 f_1} \right), \\ w_0 : (f_0, f_1, f_2) &\mapsto \left( \frac{a_0(f_0 + 1)}{b f_0 f_1}, \frac{a_0 f_0 + a_0 + b f_0 f_1}{a_0 b f_0(f_0 + 1)}, \frac{b^2 f_0}{f_2(a_0 f_0 + a_0 + b f_0 f_1)} \right), \\ w_1 : (f_0, f_1) &\mapsto (f_1, f_0), \\ \pi : (f_1, f_2) &\mapsto \left( \frac{a_0(f_0 + 1)}{b f_0 f_1}, \frac{b f_1}{a_0(f_0 + 1)} \right). \end{aligned}$$

Define  $f$ -functions by

$$f_0^{l_1, l_2, l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3}(f_0), \quad f_1^{l_1, l_2, l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3}(f_1), \quad f_2^{l_1, l_2, l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3}(f_2), \quad (4.11)$$

where  $n, N \in \mathbb{Z}$ . These form the edges of a lattice, which we refer to as the  $f$ -lattice, shown in Figure 7. The relations in the  $T_1$ -direction on the lattice:

$$T_1(f_1) = \frac{a_0(f_0 + 1)}{b f_0 f_1}, \quad T_1(f_0) = \frac{T_1(f_1) + 1}{b T_1(f_1) f_0} \quad (4.12)$$

lead to a system of first-order ordinary difference equations [20]:

$$f_1^{l_1+1, l_2, l_3} = \frac{q^{l_2-l_3} a_0(f_0^{l_1, l_2, l_3} + 1)}{q^{l_1-l_3} b f_0^{l_1, l_2, l_3} f_1^{l_1, l_2, l_3}}, \quad f_0^{l_1+1, l_2, l_3} = \frac{f_1^{l_1+1, l_2, l_3} + 1}{q^{l_1-l_3} b f_1^{l_1+1, l_2, l_3} f_0^{l_1, l_2, l_3}}, \quad (4.13)$$

or a single second-order ordinary difference equation [28–30]:

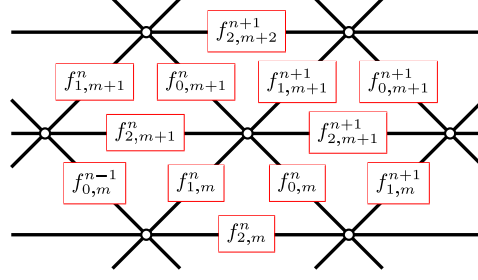
$$\left( f_0^{l_1+1, l_2, l_3} f_0^{l_1, l_2, l_3} - \frac{1}{q^{l_1-l_3} b} \right) \left( f_0^{l_1-1, l_2, l_3} f_0^{l_1, l_2, l_3} - \frac{1}{q^{l_1-l_3-1} b} \right) = \frac{q^{-l_1-l_2+2l_3} a_1}{b} \frac{f_0^{l_1, l_2, l_3}}{1 + f_0^{l_1, l_2, l_3}}, \quad (4.14)$$

which is known as a  $q$ -discrete analogue of Painlevé II equation. Moreover, from  $T_2$ - and  $T_3$ -directions we obtain the following systems of first-order ordinary difference equations:

$$\begin{cases} f_2^{l_1, l_2+1, l_3} f_2^{l_1, l_2, l_3} = \frac{q^{l_1-l_3-1} b}{f_1^{l_1, l_2, l_3} (1 + f_1^{l_1, l_2, l_3})}, \\ f_1^{l_1, l_2+1, l_3} f_1^{l_1, l_2, l_3} = \frac{q^{l_2-l_3} a_0 (q f_2^{l_1, l_2+1, l_3} + q^{l_1-l_3} b)}{f_2^{l_1, l_2+1, l_3} (q^{l_2-l_3+1} a_0 f_2^{l_1, l_2+1, l_3} + q^{l_1-l_3} b)}, \end{cases} \quad (4.15)$$

$$\begin{cases} f_0^{l_1, l_2, l_3+1} f_0^{l_1, l_2, l_3} = \frac{q f_2^{l_1, l_2, l_3} + q^{l_1-l_2} a_1 b}{f_2^{l_1, l_2, l_3} (q f_2^{l_1, l_2, l_3} + q^{l_1-l_3} b)}, \\ f_2^{l_1, l_2, l_3+1} f_2^{l_1, l_2, l_3} = \frac{q^{l_1-l_2-1} a_1 b}{f_0^{l_1, l_2, l_3+1} (f_0^{l_1, l_2, l_3+1} + 1)}, \end{cases} \quad (4.16)$$



Figure 7. Configuration of  $f$ -functions on the  $f$ -lattice.

respectively.

Finally, we consider the  $\omega$ -lattice. Letting

$$\omega_0 = \frac{\tau_{-1}}{\tau_{-3}}, \quad \omega_1 = \frac{\tau_1}{\tau_{-1}}, \quad \omega_2 = \frac{\tau_0}{\tau_{-2}}, \quad (4.17)$$

we obtain the action of  $\widetilde{W}((A_1 + A'_1)^{(1)})$  on the variables  $\omega_i$ :

$$\begin{aligned} s_0 : (\omega_0, \omega_1) &\mapsto \left( \frac{a_0\omega_0(\omega_2^2 + \omega_0\omega_2 + \omega_0\omega_1)}{a_0\omega_0\omega_1 + \omega_2^2 + \omega_0\omega_2}, \frac{\omega_1(a_0\omega_0\omega_2 + a_0\omega_0\omega_1 + \omega_2^2)}{\omega_2^2 + \omega_0\omega_2 + \omega_0\omega_1} \right), \\ s_1 : \omega_2 &\mapsto \frac{a_1\omega_2(a_0\omega_0 + b\omega_1)}{a_0a_1\omega_0 + b\omega_1}, \\ w_0 : (\omega_0, \omega_1, \omega_2) &\mapsto \left( \frac{b^2\omega_1}{a_0\omega_0\omega_1 + a_0\omega_0\omega_2 + b\omega_2\omega_1}, \frac{1}{\omega_1}, \frac{b\omega_2}{a_0(\omega_0\omega_1 + \omega_0\omega_2)} \right), \\ w_1 : (\omega_0, \omega_1, \omega_2) &\mapsto \left( \frac{1}{\omega_1}, \frac{1}{\omega_0}, \frac{1}{\omega_2} \right), \\ \pi : (\omega_0, \omega_1, \omega_2) &\mapsto \left( \frac{b\omega_2}{a_0(\omega_0\omega_1 + \omega_0\omega_2)}, \frac{1}{\omega_2}, \frac{1}{\omega_1} \right). \end{aligned}$$

We define  $\omega$ -functions by

$$\omega_{l_1, l_2, l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3}(\omega_0), \quad (4.18)$$

where  $l_1, l_2, l_3 \in \mathbb{Z}$  and the  $\omega$ -lattice is as shown in Figure 8. We note that

$$\omega_0 = \omega_{0,0,0}, \quad \omega_1 = \omega_{1,0,0}, \quad \omega_2 = \omega_{1,1,0}. \quad (4.19)$$

**Lemma 4.2.** *Since for all  $w \in \widetilde{W}((A_1 + A'_1)^{(1)})$ ,*

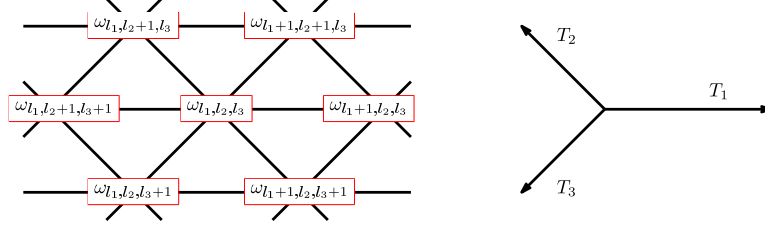
$$w(\omega_i) \in \mathcal{L} \quad (i = 0, 1, 2), \quad (4.20)$$

where  $\mathcal{L} = \mathcal{K}(\omega_0, \omega_1, \omega_2)$  is the field of rational functions in  $\omega_i$ ,  $i = 0, 1, 2$ , with coefficient field  $\mathcal{K} = \mathbb{C}(a_0, a_1, b)$ , every point on the  $\omega$ -lattice is determined by three initial points. This implies that quad-equations appear the relations on the  $\omega$ -lattice. Moreover, relations on the  $f$ -lattice can be expressed by those on the  $\omega$ -lattice because of the following correspondence:

$$f_0 = \frac{\omega_1}{\omega_2}, \quad \left( \text{or } f_0^{l_1, l_2, l_3} = \frac{\omega_{l_1+1, l_2, l_3}}{\omega_{l_1+1, l_2+1, l_3}} \right), \quad (4.21a)$$

$$f_1 = \frac{\omega_2}{\omega_0}, \quad \left( \text{or } f_1^{l_1, l_2, l_3} = \frac{\omega_{l_1+1, l_2+1, l_3}}{\omega_{l_1, l_2, l_3}} \right), \quad (4.21b)$$

$$f_2 = \frac{\omega_0}{\omega_1}, \quad \left( \text{or } f_2^{l_1, l_2, l_3} = \frac{\omega_{l_1, l_2, l_3}}{\omega_{l_1+1, l_2, l_3}} \right). \quad (4.21c)$$

Figure 8. Configuration of  $\omega$ -functions on the  $\omega$ -lattice.

To show the relation between the  $\omega$ -lattice and a cube associated with ABS equations, we derive relations (4.22) on the  $\omega$ -lattice.

**Lemma 4.3.** *The following equations hold on the  $\omega$ -lattice:*

$$\frac{\omega_{l_1+1, l_2, l_3+1}}{\omega_{l_1, l_2, l_3}} - q^{l_1-l_2} \frac{b}{a_0} \frac{\omega_{l_1+1, l_2, l_3}}{\omega_{l_1, l_2, l_3+1}} = -1, \quad (4.22a)$$

$$\frac{\omega_{l_1+1, l_2+1, l_3}}{\omega_{l_1, l_2, l_3}} - q^{l_1-l_3-1} b \frac{\omega_{l_1+1, l_2, l_3}}{\omega_{l_1, l_2+1, l_3}} = -1, \quad (4.22b)$$

$$\frac{\omega_{l_1, l_2+1, l_3+1}}{\omega_{l_1, l_2, l_3}} = q^{l_1-l_2-1} \frac{b}{a_0} \frac{\omega_{l_1, l_2+1, l_3}}{\omega_{l_1, l_2, l_3+1} - \omega_{l_1, l_2+1, l_3}} - q^{l_2-l_3-1} a_0 \omega_{l_1, l_2, l_3+1}. \quad (4.22c)$$

We note that Equations (4.22a) and (4.22b) are  $D4_{(\delta_1, \delta_2, \delta_3)=(1,0,0)}$ -type equations, while Equation (4.22c) is a  $H3_{(\delta, \epsilon)=(0,0)}$ -type equation.

*Proof.* From the action of translations  $T_1$ ,  $T_2$  and  $T_3$ , it follows that

$$\frac{\omega_1}{\omega_2} - \frac{b}{a_0} \frac{T_1(\omega_2)}{\omega_0} = -1, \quad (4.23a)$$

$$\frac{\omega_2}{\omega_0} - q^{-1} b \frac{\omega_1}{T_2(\omega_0)} = -1, \quad (4.23b)$$

$$\frac{\omega_0}{\omega_1} = \frac{b}{a_0} \frac{\omega_2 - q^{-1} a_0 T_3(\omega_1)}{T_3(\omega_1) - \omega_2}. \quad (4.23c)$$

Applying  $T_1^{l_1} T_2^{l_2} T_3^{l_3+1}$ ,  $T_1^{l_1} T_2^{l_2} T_3^{l_3}$  and  $T_1^{l_1} T_2^{l_2+1} T_3^{l_3+1}$  on Equations (4.23a)–(4.23c), we obtain Equations (4.22a)–(4.22c), respectively. This completes the proof.  $\square$

In [15], we show that the following proposition follows from Lemma 4.3.

**Proposition 4.4** ([15]). *The  $\omega$ -lattice can be obtained from an asymmetric 3D cube which has two  $H3_{(\delta, \epsilon)=(0,0)}$ -type equations and four  $D4_{(\delta_1, \delta_2, \delta_3)=(1,0,0)}$ -type equations associated with each face.*

It follows from Proposition 4.4 that the above quad-equations are the only ones that relate four points on the  $\omega$ -variables. This provides a part of Theorem 1.1.

In a similar manner as the case  $(A_2 + A_1)^{(1)}$ , we can also obtain the discrete Schwarzian KdV equation from the  $\omega$ -lattice. Actually, letting

$$z_{l_1, l_2, l_3} = T_1^{l_1} T_2^{l_2} T_3^{l_3}(z), \quad (4.24)$$

where

$$z = \omega_0 \omega_1, \quad (4.25)$$

we obtain

$$\frac{(z_{l_1, l_2, l_3} - q^{-l_1+l_3+1}b^{-1}z_{l_1, l_2+1, l_3})(z_{l_1+1, l_2, l_3} - q^{-l_1+l_3}b^{-1}z_{l_1+1, l_2+1, l_3})}{z_{l_1, l_2, l_3}z_{l_1, l_2+1, l_3}} = q^{2(-l_1+l_3)+1}b^{-2}, \quad (4.26a)$$

$$\frac{(z_{l_1, l_2, l_3} - q^{-l_1+l_2}a_0b^{-1}z_{l_1, l_2, l_3+1})(z_{l_1+1, l_2, l_3} - q^{-l_1+l_2-1}a_0b^{-1}z_{l_1+1, l_2, l_3+1})}{z_{l_1, l_2, l_3}z_{l_1, l_2, l_3+1}} = q^{2(-l_1+l_2)-1}a_0^2b^{-2}, \quad (4.26b)$$

$$\frac{(z_{l_1, l_2, l_3} - q^{-l_1+l_2}a_0b^{-1}z_{l_1, l_2, l_3+1})(z_{l_1, l_2+1, l_3} - q^{-l_1+l_2+1}a_0b^{-1}z_{l_1, l_2+1, l_3+1})}{(z_{l_1, l_2, l_3} - q^{-l_1+l_3+1}b^{-1}z_{l_1, l_2+1, l_3})(z_{l_1, l_2, l_3+1} - q^{-l_1+l_3+2}b^{-1}z_{l_1, l_2+1, l_3+1})} = q^{2(l_2-l_3)}a_1^{-2}, \quad (4.26c)$$

from the following relations:

$$\frac{(z - qb^{-1}T_2(z))(T_1(z) - b^{-1}T_1T_2(z))}{zT_2(z)} = qb^{-2}, \quad (4.27a)$$

$$\frac{(z - a_0b^{-1}T_3(z))(T_1(z) - q^{-1}a_0b^{-1}T_1T_3(z))}{zT_3(z)} = q^{-1}a_0^2b^{-2}, \quad (4.27b)$$

$$\frac{(z - a_0b^{-1}T_3(z))(T_2(z) - qa_0b^{-1}T_2T_3(z))}{(z - qb^{-1}T_2(z))(T_3(z) - q^2b^{-1}T_2T_3(z))} = a_1^{-2}, \quad (4.27c)$$

respectively. We note that Equations (4.26a) and (4.26b) are  $H1_{\epsilon=0}$ -type equations, while Equation (4.26c) is a  $Q1_{\epsilon=0}$ -type equation.

**4.2. The restricted cases.** In order to consider the restricted cases, we introduce the half-translation  $R_1$  defined by

$$R_1 = \pi w_1, \quad (4.28)$$

which satisfies

$$R_1^2 = T_1. \quad (4.29)$$

The action of  $R_1$  on the parameters is given by

$$R_1 : (a_0, a_1, b) \mapsto (a_1, a_0, a_1b),$$

while its action on variables are given by

$$R_1 : (\tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1) \mapsto (\tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2),$$

$$R_1 : (f_0, f_1) \mapsto \left( \frac{a_0(f_0 + 1)}{bf_0f_1}, f_0 \right),$$

$$R_1 : (\omega_0, \omega_1, \omega_2) \mapsto \left( \omega_2, \frac{a_0\omega_0(\omega_1 + \omega_2)}{b\omega_2}, \omega_1 \right).$$

The restricted functions are defined by

$$\tau^{(l)} = R_1^l(\tau_{-3}), \quad f^{(l)} = R_1^l(f_0), \quad \omega^{(l)} = R_1^l(\omega_0), \quad (4.30)$$

where

$$\tau^{(2l)} = \tau_0^l, \quad \tau^{(2l+1)} = \tau_1^{l+1}, \quad (4.31a)$$

$$f^{(2l)} = f_0^{l,0,0}, \quad f^{(2l+1)} = f_1^{l+1,0,0}, \quad (4.31b)$$

$$\omega^{(2l)} = \omega_{l,0,0}, \quad \omega^{(2l+1)} = \omega_{l+1,1,0}. \quad (4.31c)$$

System (4.13) on the restricted  $f$ -lattice where  $f^{(l)}$  are defined can be rewritten as the following single equation

$$f^{(l+1)}f^{(l-1)} = \frac{R_1^l(a_0)(f^{(l)} + 1)}{R_1^l(b)f^{(l)}}. \quad (4.32)$$

We note that when

$$a_0 = q^{1/2}, \quad (4.33)$$

transformation  $R_1$  becomes the translational motion in the parameter subspace:

$$R_1 : b \mapsto q^{1/2}b, \quad (4.34)$$

and then Equation (4.32) can be regarded as the single second-order ordinary difference equation:

$$f^{(l+1)} f^{(l-1)} = \frac{f^{(l)} + 1}{q^{(l-1)/2} b f^{(l)}}, \quad (4.35)$$

which is known as a  $q$ -discrete analogue of Painlevé I equation [28].

Furthermore, Equations (4.22a) and (4.22b) on the restricted  $\omega$ -lattice where  $\omega^{(l)}$  are defined can be expressed by the following single equation:

$$\frac{\omega^{(l+3)}}{\omega^{(l)}} - R_1^l \left( \frac{a_0}{b} \right) \frac{\omega^{(l+2)}}{\omega^{(l+1)}} = R_1^l \left( \frac{a_0}{b} \right). \quad (4.36)$$

We note that Equations (4.26a) and (4.26b) on the restricted  $\omega$ -lattice can be expressed by the following single equation:

$$\frac{(z^{(l)} - R_1^l(a_0^{-1}b)z^{(l+1)})(z^{(l+2)} - R_1^l(a_1b)z^{(l+3)})}{z^{(l)}z^{(l+1)}} = 1, \quad (4.37)$$

where

$$z^{(l)} = R_1^l(z). \quad (4.38)$$

In a similar manner as the case  $(A_2 + A_1)^{(1)}$ , from periodic reductions of partial difference equations, we can obtain the quad-equations on the restricted  $\omega$ -lattice as the following lemma.

**Lemma 4.5.** *Equations (4.36) and (4.37) can be respectively obtained by periodic reductions of  $D4_{(\delta_1, \delta_2, \delta_3)=(1,0,0)}$ - and  $H1_{\epsilon=0}$ -type equations:*

$$\begin{aligned} D4(-\alpha\beta^{-1}U, \widehat{\overline{U}}, \overline{\widehat{U}}, \widehat{\overline{U}}; 1, 0, 0) &= 0 \\ \Leftrightarrow \frac{\widehat{\overline{U}}}{\overline{U}} - \frac{\alpha \overline{U}}{\beta \widehat{\overline{U}}} &= \frac{\alpha}{\beta}, \end{aligned} \quad (4.39)$$

$$\begin{aligned} H1(U^{-1}, \overline{\alpha\beta\widehat{\overline{U}}}, \alpha^{-1}\beta^{-1}\widehat{\overline{U}}^{-1}, \overline{U}; \alpha^{-2}, \alpha^{-1}(\alpha^{-1} - \beta^{-1}); 0) &= 0 \\ \Leftrightarrow \frac{(U - \alpha\beta\widehat{\overline{U}})(\overline{U} - \overline{\alpha\beta\widehat{\overline{U}}})}{U\widehat{\overline{U}}} &= 1, \end{aligned} \quad (4.40)$$

where we have used the notation (1.8), with the  $(1, -2)$ -periodic condition

$$U_{l+1, m-2} = U_{l, m}. \quad (4.41)$$

*Proof.* Equation (4.39) and periodic condition (4.41) imply the condition on the parameters

$$\frac{\overline{\alpha}}{\alpha} = \frac{\beta}{\widehat{\overline{\beta}}} = q^{-1}. \quad (4.42)$$

Therefore, Equation (4.39) can be reduced to

$$\frac{\widehat{\widehat{\overline{U}}}}{\overline{U}} - \frac{\alpha \widehat{\overline{U}}}{\beta \widehat{\overline{U}}} = \frac{\alpha}{\beta}. \quad (4.43)$$

Then, the statement holds since Equation (4.43) is equivalent to Equation (4.36) with the following correspondence:

$$U_{0,0} = \omega^{(0)}, \quad \frac{\alpha_0}{\beta_0} = \frac{a_0}{b}, \quad \overline{\phantom{x}} = T_1, \quad \widehat{\phantom{x}} = R_1. \quad (4.44)$$

In a similar manner, we obtain Equation (4.37) from the reduction of Equation (4.40) with the following correspondence:

$$U_{0,0} = z^{(0)}, \quad \alpha_0 \beta_0 = \frac{b}{a_0}, \quad \frac{\bar{\alpha}}{\alpha} = q, \quad \bar{\tau} = T_1, \quad \hat{\tau} = R_1. \quad (4.45)$$

Therefore, we have completed the proof.  $\square$

## 5. CONCLUDING REMARKS

In this paper, we constructed  $\omega$ -lattices associated with the extended affine Weyl groups of types  $(A_2 + A_1)^{(1)}$  and  $(A_1 + A_1)^{(1)}$ .

More general  $\omega$ -lattices are possible. They share certain fundamental properties with the  $\omega$ -lattice constructed in Section 3. In particular, all  $f$ -functions arise as rational combinations of  $\omega$ -functions and all  $\omega$ -functions in each connected component of an  $\omega$ -lattice are determined by three initial variables in that component. We will explore the general constructions of  $\omega$ -lattices in subsequent works. An interesting future project is to construct various  $\omega$ -lattices associated with Painlevé systems of other surface types in Sakai's classification [30].

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## APPENDIX A. PROOF OF LEMMA 4.1

In this section, we prove Lemma 4.1.

We consider the following eight base points:

$$P_1 : (f, g) = (-1, 0), \quad (A.1a)$$

$$P_2 : (f, g) = (0, -1), \quad (A.1b)$$

$$P_3 : (f, g) = (\infty, 0), \quad P_6 : (f, g; fg) = (\infty, 0; -q^{-1}b^{-1}), \quad (A.1c)$$

$$P_4 : (f, g) = (0, \infty), \quad P_7 : (f, g; fg) = (0, \infty; -a_0b^{-1}), \quad (A.1d)$$

$$P_5 : (f, g) = (\infty, \infty), \quad P_8 : (f, g; f/g) = (\infty, \infty; -a_0^{-1}), \quad (A.1e)$$

where  $a_0, b, q$  are complex parameters. These base points are of the following  $q$ -difference equation:

$$\bar{g} = \frac{a_0(f+1)}{bfg}, \quad \bar{f} = \frac{\bar{g}+1}{b\bar{g}f}, \quad (A.2)$$

where  $\bar{\cdot}$  means  $t \mapsto qt$ , which is equivalent to System (4.13). Let  $\epsilon : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  denotes blow up of  $\mathbb{P}^1 \times \mathbb{P}^1$  at the points (A.1). The linear equivalence classes of the total transform of the coordinate lines  $f=\text{constant}$  and  $g=\text{constant}$  are denoted by  $H_f$  and  $H_g$ , respectively. The Picard group of  $X$ , denoted by  $\text{Pic}(X)$ , is given by

$$\text{Pic}(X) = \mathbb{Z}H_f \oplus \mathbb{Z}H_g \oplus \bigoplus_{i=1}^8 \mathbb{Z}E_i, \quad (A.3)$$

where  $E_i = \epsilon^{-1}(P_i)$  is the total transform of the point of the  $i$ -th blow up. The intersection form  $(\cdot | \cdot)$  is defined by

$$(H_f | H_g) = 1, \quad (H_f | H_f) = (H_g | H_g) = (H_f | E_i) = (H_g | E_i) = 0, \quad (E_i | E_j) = -\delta_{ij}, \quad (A.4)$$

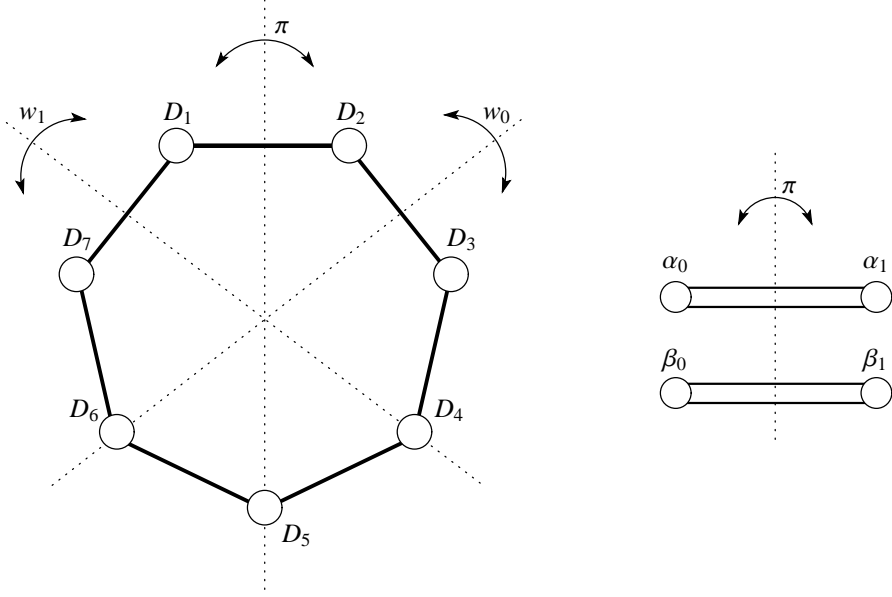


Figure 9. Dynkin diagrams for the root lattices. Left:  $Q(A_6^{(1)})$ , right:  $Q(A_6^{(1)\perp})$ .

where  $1 \leq i \leq 8$ ,  $1 \leq j \leq 8$  are integers. The anti-canonical divisor of  $X$ , denoted by  $-K_X$ , is uniquely decomposed into the prime divisors:

$$\delta = -K_X = 2H_f + 2H_g - \sum_{i=1}^8 E_i = \sum_{i=1}^7 D_i,$$

where

$$D_1 = H_f - E_2 - E_4, \quad D_2 = E_4 - E_7, \quad D_3 = H_g - E_4 - E_5, \quad D_4 = E_5 - E_8, \quad (\text{A.5a})$$

$$D_5 = H_f - E_3 - E_5, \quad D_6 = E_3 - E_6, \quad D_7 = H_g - E_1 - E_3. \quad (\text{A.5b})$$

We can show that the corresponding Cartan matrix

$$(d_{ij})_{i,j=1}^7 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad (\text{A.6})$$

where

$$d_{ij} = \frac{2(D_i|D_j)}{(D_j|D_j)}, \quad (\text{A.7})$$

and Dynkin diagram (see Figure 9) are of type  $A_6^{(1)}$ . Thus, we can set the root lattice as

$$Q(A_6^{(1)}) = \bigoplus_{i=1}^7 \mathbb{Z}D_i, \quad (\text{A.8})$$

and identify the surface  $X$  as being type  $A_6^{(1)}$  in Sakai's list.

Moreover, we obtain the orthogonal root lattice

$$Q(A_6^{(1)\perp}) = \mathbb{Z}\alpha_0 \bigoplus \mathbb{Z}\alpha_1 \bigoplus \mathbb{Z}\beta_0 \bigoplus \mathbb{Z}\beta_1, \quad (\text{A.9})$$

where

$$\alpha_0 = H_f + H_g - E_1 - E_2 - E_5 - E_8, \quad (\text{A.10a})$$

$$\alpha_1 = H_f + H_g - E_3 - E_4 - E_6 - E_7, \quad (\text{A.10b})$$

$$\beta_0 = 3H_f + H_g - 3E_1 + E_2 - 2E_4 - E_5 - 2E_7 - E_8, \quad (\text{A.10c})$$

$$\beta_1 = -H_f + H_g + 2E_1 - 2E_2 - E_3 + E_4 - E_6 + E_7, \quad (\text{A.10d})$$

$$\delta = \alpha_0 + \alpha_1 = \beta_0 + \beta_1, \quad (\text{A.10e})$$

by searching for elements of  $\text{Pic}(X)$  that are orthogonal to all divisors  $D_i$ ,  $i = 1, \dots, 7$ . The root lattice  $Q(A_6^{(1)\perp})$  can be divided into the following two root lattice

$$\mathbb{Z}\alpha_0 \bigoplus \mathbb{Z}\alpha_1, \quad \mathbb{Z}\beta_0 \bigoplus \mathbb{Z}\beta_1, \quad (\text{A.11})$$

since  $\alpha_i$ ,  $i = 0, 1$ , and  $\beta_i$ ,  $i = 0, 1$ , are orthogonal to each other:  $(\alpha_i|\beta_j) = 0$  where  $i = 0, 1$  and  $j = 0, 1$ . Moreover, their corresponding Cartan matrices

$$(a_{ij})_{i,j=0}^1 = (b_{ij})_{i,j=0}^1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad (\text{A.12})$$

where

$$a_{ij} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}, \quad b_{ij} = \frac{2(\beta_i|\beta_j)}{(\beta_j|\beta_j)}, \quad (\text{A.13})$$

and Dynkin diagram (see Figure 9) are of type  $A_1^{(1)}$ . Therefore, we can set the root lattices as follows:

$$Q(A_{1,|\alpha|^2=2}^{(1)}) = \mathbb{Z}\alpha_0 \bigoplus \mathbb{Z}\alpha_1, \quad Q(A_{1,|\beta|^2=14}^{(1)}) = \mathbb{Z}\beta_0 \bigoplus \mathbb{Z}\beta_1, \quad (\text{A.14})$$

Let us consider the Cremona isometries for this setting. A Cremona isometry is defined by an automorphism of  $\text{Pic}(X)$  which preserves

- (i): the intersection form on  $\text{Pic}(X)$ ;
- (ii): the canonical divisor  $K_X$ ;
- (iii): effectiveness of each effective divisor of  $\text{Pic}(X)$ .

It is well known that automorphisms of the Dynkin diagram corresponding to the divisors and reflections for simple roots which orthogonal to all divisors are Cremona isometries and form (extended) affine Weyl group [6, 21, 30]. We define the reflections  $s_i$ ,  $i = 0, 1$ , respectively across the hyperplane orthogonal to the root  $\alpha_i$ ,  $i = 0, 1$ , by

$$s_i(v) = v - \frac{2(v|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i \quad (\text{A.15})$$

for all  $v \in \text{Pic}(X)$ . We can easily verify that the actions of  $W(A_1^{(1)}) = \langle s_0, s_1 \rangle$  on  $\text{Pic}(X)$  satisfy the fundamental relations of the affine Weyl group of type  $A_1^{(1)}$ :

$$s_0^2 = s_1^2 = (s_0 s_1)^\infty = 1. \quad (\text{A.16})$$

Note that the reflections corresponding to the roots  $\beta_i$ ,  $i = 0, 1$ , cannot be constructed by this way since their self-intersection numbers are  $-14$ . We also define the group of the

diagram automorphisms  $\text{Aut}(A_6^{(1)}) = \langle w_0, w_1, \pi \rangle$  by

$$w_0 : \begin{pmatrix} H_f \\ H_g \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} H_f \\ H_g \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \end{pmatrix}, \quad (\text{A.17a})$$

$$w_1 : \begin{pmatrix} H_f \\ H_g \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} H_f \\ H_g \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \end{pmatrix}, \quad (\text{A.17b})$$

$$\pi : \begin{pmatrix} H_f \\ H_g \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} H_f \\ H_g \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \end{pmatrix}, \quad (\text{A.17c})$$

where their actions on the divisors are given by

$$w_0 : (D_1, D_2, D_3, D_4, D_5, D_6, D_7) \mapsto (D_4, D_3, D_2, D_1, D_7, D_6, D_5), \quad (\text{A.18a})$$

$$w_1 : (D_1, D_2, D_3, D_4, D_5, D_6, D_7) \mapsto (D_7, D_6, D_5, D_4, D_3, D_2, D_1), \quad (\text{A.18b})$$

$$\pi : (D_1, D_2, D_3, D_4, D_5, D_6, D_7) \mapsto (D_2, D_1, D_7, D_6, D_5, D_4, D_3). \quad (\text{A.18c})$$

Since transformations  $w_i$ ,  $i = 0, 1$ , respectively correspond to the reflections of the roots  $\beta_i$ ,  $i = 0, 1$ , as follows:

$$w_0 : (\alpha_0, \alpha_1, \beta_0, \beta_1) \mapsto (\alpha_0, \alpha_1, -\beta_0, \beta_1 + 2\beta_0), \quad (\text{A.19a})$$

$$w_1 : (\alpha_0, \alpha_1, \beta_0, \beta_1) \mapsto (\alpha_0, \alpha_1, \beta_0 + 2\beta_1, -\beta_1), \quad (\text{A.19b})$$

and satisfy the fundamental relations of the affine Weyl group of type  $A_1^{(1)}$ :

$$w_0^2 = w_1^2 = (w_0 w_1)^\infty = 1, \quad (\text{A.20})$$

we can set  $W(A_{1, |\beta|^2=14}^{(1)}) = \langle w_0, w_1 \rangle$ . Note that the action of  $W(A_1^{(1)}) = \langle s_0, s_1 \rangle$  and that of  $W(A_{1, |\beta|^2=14}^{(1)}) = \langle w_0, w_1 \rangle$  commute. Moreover, since the action of  $\pi$  on the roots are given by

$$\pi : (\alpha_0, \alpha_1, \beta_0, \beta_1) \mapsto (\alpha_1, \alpha_0, \beta_1, \beta_0), \quad (\text{A.21})$$



we can set  $\text{Aut}((A_1 + A_{1,|\beta|^2=14})^{(1)}) = \langle \pi \rangle$ . Note that the transformation  $\pi$  satisfies the following relations

$$\pi^2 = 1, \quad \pi s_0 = s_1 \pi, \quad \pi w_0 = w_1 \pi. \quad (\text{A.22})$$

Therefore, the group of Cremona isometries

$$W(A_1^{(1)}) \rtimes \text{Aut}(A_6^{(1)}) = W((A_1 + A_{1,|\beta|^2=14})^{(1)}) \rtimes \text{Aut}((A_1 + A_{1,|\beta|^2=14})^{(1)}) \quad (\text{A.23})$$

form the extended affine Weyl group of type  $(A_1 + A_1)^{(1)}$ , denoted by  $\widetilde{W}((A_1 + A_1')^{(1)})$ .

Finally, we construct the  $\tau$  functions after [16, 32–34]. We define the variables  $f_u, f_d, g_u$  and  $g_d$  by

$$f = \frac{f_u}{f_d}, \quad g = \frac{g_u}{g_d}, \quad (\text{A.24})$$

and their polynomial  $F_\Lambda$  by

$$F_\Lambda = F_\Lambda(f_u, f_d, g_u, g_d), \quad (\text{A.25})$$

where  $\Lambda = mH_f + nH_g - \sum_{i=1}^8 \mu_i E_i$  which corresponds to a curve of bi-degree  $(m, n)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  passing through base points  $P_i$  with multiplicity  $\mu_i$ . For example,

$$F_{H_f+H_g-E_2-E_5-E_8} = \gamma(a_0 f_u g_d + f_d g_u + f_d g_d), \quad (\text{A.26})$$

where  $\gamma$  is an arbitrary non-zero complex parameter.

**Definition A.1.** We define a mapping  $\tau$  on the set

$$M = \{w(E_i) \mid w \in \widetilde{W}((A_1 + A_1')^{(1)}), i = 1, \dots, 8\} \quad (\text{A.27})$$

by the following conditions:

(i):

$$w.\tau(\Lambda) = \tau(w^{-1}(\Lambda)), \quad (\text{A.28})$$

where  $w \in \widetilde{W}((A_1 + A_1')^{(1)})$  and  $\Lambda \in M$ ;

(ii):

$$\tau(\Lambda) = \frac{F_\Lambda(f_u, f_d, g_u, g_d)}{\tau(E_1)^{\mu_1} \cdots \tau(E_8)^{\mu_8}}, \quad (\text{A.29})$$

for  $\Lambda = mH_f + nH_g - \sum_{i=1}^8 \mu_i E_i \in M$ ;

(iii):

$$\frac{F_\Lambda(f_u, f_d, g_u, g_d)}{F_\Lambda(1, 1, 1, 1)} = \tau(E_1)^{\mu_1} \cdots \tau(E_8)^{\mu_8}, \quad (\text{A.30})$$

for  $\Lambda = mH_f + nH_g - \sum_{i=1}^8 \mu_i E_i \in \{D_1, D_3, D_5, D_7\}$ , which are equivalent to

$$f_u = \tau(E_2)\tau(E_4), \quad f_d = \tau(E_3)\tau(E_5), \quad g_u = \tau(E_1)\tau(E_3), \quad g_d = \tau(E_4)\tau(E_5). \quad (\text{A.31})$$

Finally, setting

$$\tau_{-3} = \tau(E_1), \quad \tau_{-2} = \tau(E_4) = \tau(E_7), \quad \tau_{-1} = \tau(E_5) = \tau(E_8), \quad (\text{A.32a})$$

$$\tau_0 = \tau(E_3) = \tau(E_6), \quad \tau_1 = \tau(E_2) \quad (\text{A.32b})$$

and normalizing the polynomials  $F_\Lambda$  to be designed to hold the fundamental relations (4.3), we have completed the proof of Lemma 4.1.

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